Integral form of solution for Shrödinger equation

So far we were avoiding inner scattering region where we have to account for potential V (and we did learn quite a lot)
But let's face the real problem

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$$

$$\Rightarrow \quad \underbrace{(\nabla^2\psi + k^2\psi)\psi}_{\nabla^2\psi + k^2\psi = 0} = \frac{2m}{\hbar^2}V\psi = Q \qquad , \quad k^2 = \frac{2mE}{\hbar^2}$$

called Helmholtz equation
the one which governs light propagation
because of 0 it is called source free
i.e. light propagation in vacuum

There is a 'trick' for solving above equation with source (Q)

Suppose we can solve

$$(\nabla^2 + k^2)\, G(\vec{r}) = \delta(\vec{r}) \qquad (*)$$

then $$\boxed{\psi(\vec{r}) = \int G(\vec{r}-\vec{r}\,')\, Q(\vec{r}\,')\, d^3\vec{r}\,'} \qquad (**)$$

Let's prove it

$$(\nabla^2 + k^2)\psi = \int \Big[\overbrace{\underbrace{(\nabla^2 + k^2)}_{\text{does not depend on } \vec{r}\,', \text{ so we can move it inside of integral}} G(\vec{r}-\vec{r}\,')}^{\delta(\vec{r}-\vec{r}\,') \text{ see } (*)} \Big] Q(r')\, d^3r' = Q(r)$$

$G(\vec{r})$ is called **the Green's function**

and the real trick is to find it.

$$\boxed{G(\vec{r}) = -\frac{e^{ikr}}{4\pi r}}$$

See Griffiths Ch 10.4.1 for this derivation

The rest is easy we plug it to $(**)$

We get

$$\psi(\vec{r}) = \psi_0 + \int \frac{e^{ik|\vec{r}-\vec{r}'|}}{4\pi \cdot |\vec{r}-\vec{r}'|} Q(r') \, dr' =$$

$$\boxed{\psi(\vec{r}) = \psi_0(\vec{r}) - \frac{2m}{\hbar^2} \int \frac{e^{ik|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|} V(\vec{r}') \psi(\vec{r}') \, d^3\vec{r}'}$$

, where ψ_0 satisfies source free equation $(\nabla^2 + k^2)\psi_0(\vec{r}) = 0$

There is "slight" problem
$\psi(\vec{r})$ is in LHS and RHS.
So this integral solution is not a solution
but yet one more equation to solve. ☹