Identical particles: exchange forces

$$\psi(x_1, x_2) = \begin{cases} \psi_a(x_1)\,\psi_b(x_2) = \psi_d & : \text{distinguishable} \\ \frac{1}{\sqrt{2}}\left(\psi_a(x_1)\,\psi_b(x_2) + \psi_a(x_2)\,\psi_b(x_1)\right) = \psi_+ & : \text{identical bosons} \\ \frac{1}{\sqrt{2}}\left(\psi_a(x_1)\,\psi_b(x_2) - \psi_a(x_2)\,\psi_b(x_1)\right) = \psi_- & : \text{identical fermions} \end{cases}$$

Our goal is to find average distance between 2 particles $\langle (x_1 - x_2)^2 \rangle$

$$\langle (x_1 - x_2)^2 \rangle = \langle x_1^2 \rangle + \langle x_2^2 \rangle - 2 \langle x_1 x_2 \rangle$$

Case 1 distinguishable particles

$$\langle x_1^2 \rangle_d = \langle \psi_d | x_1^2 | \psi_d \rangle = \underbrace{\int x_1^2 |\psi_a(x_1)|^2 dx_1}_{\langle x^2 \rangle_a} \cdot \underbrace{\int |\psi_b|^2 dx_2}_{=1} = \langle x^2 \rangle_a$$

similarly $\langle x_2^2 \rangle = \langle x^2 \rangle_b$

$$\langle x_1 \cdot x_2 \rangle_d = \underbrace{\int x_1 |\psi_a(x_1)|^2 dx_1}_{\langle x \rangle_a} \cdot \underbrace{\int x_2 |\psi_b(x_2)|\, dx_2}_{\langle x \rangle_b} = \langle x \rangle_a \langle x \rangle_b$$

$$\langle (x_1 - x_2)^2_d \rangle = \boxed{\langle (\Delta x)^2 \rangle_d = \langle x^2 \rangle_a + \langle x^2 \rangle_b - 2\langle x \rangle_a \langle x \rangle_b}$$

identical particles case:

$$\begin{aligned}
\langle x_1^2 \rangle_\pm = \frac{1}{2}\Big[& \underbrace{\int x_1^2 |\psi_a(x_1)|^2 dx_1}_{\langle x^2 \rangle_a} \cdot \underbrace{\int |\psi_b(x_2)|^2 dx_2}_{=1} + \\
+ & \underbrace{\int x_1^2 |\psi_b(x_1)|^2 dx_1}_{\langle x^2 \rangle_b} \cdot \underbrace{\int |\psi_a(x_2)|^2 dx_2}_{=1} \\
\pm & \int x_1^2 \psi_a^*(x_1)\psi_b(x_1) dx_1 \cdot \underbrace{\int \psi_b^*(x_2)\psi_a(x_2) dx_2}_{=0} \\
\pm & \int x_1^2 \psi_b^*(x_1)\psi_a(x_1) dx_1 \cdot \underbrace{\int \psi_a^*(x_2)\psi_b(x_2) dx_2}_{=0} \Big]
\end{aligned}$$

notice that subindex 1 is gone after integration

$$= \frac{1}{2}\left[\langle x^2 \rangle_a + \langle x^2 \rangle_b \right] = \langle x_2^2 \rangle_\pm$$

due to symmetry plus we cannot tell particles 1 and 2 apart next

$$\langle x_1 x_2\rangle_\pm = \frac{1}{2}\Big[\underbrace{\int x_1 |\psi_a(x_1)|^2 dx_1}_{\langle x\rangle_a} \cdot \underbrace{\int x_2 |\psi_b(x_2)|^2 dx_2}_{\langle x\rangle_b} +$$

$$+ \underbrace{\int x_1 |\psi_b(x_1)|^2 dx_1}_{\langle x\rangle_b} \cdot \underbrace{\int x_2 |\psi_a(x_2)|^2 dx_1}_{\langle x\rangle_a} \pm$$

$$\pm \underbrace{\int x_1 \psi_a^*(x_1)\psi_b(x_1)\,dx_1}_{\langle x\rangle_{ab}} \cdot \underbrace{\int x_2 \psi_b^*(x_2)\psi_a(x_2)\,dx_2}_{\langle x\rangle_{ba}}$$

$$\pm \underbrace{\int x_1 \psi_b^*(x_1)\psi_a(x_1)\,dx_1}_{\langle x\rangle_{ba}} \cdot \underbrace{\int x_2 \psi_a^*(x_2)\psi_b(x_2)\,dx_2}_{\langle x\rangle_{ab}}$$

$$= \langle x\rangle_a \langle x\rangle_b \pm \underbrace{\langle x\rangle_{ab}\langle x\rangle_{ba}}_{|\langle x\rangle_{ab}|^2} \implies$$

$$\langle (x_1 - x_2)^2\rangle_\pm = \underbrace{\langle x^2\rangle_a + \langle x^2\rangle_b - 2\langle x\rangle_a\langle x\rangle_b}_{\langle(\Delta x)^2\rangle_d} \mp 2|\langle x\rangle_{ab}|^2$$

$$\boxed{\langle(\Delta x)^2\rangle_\pm = \langle(\Delta x)^2\rangle_d \mp 2|\langle x\rangle_{ab}|^2}$$

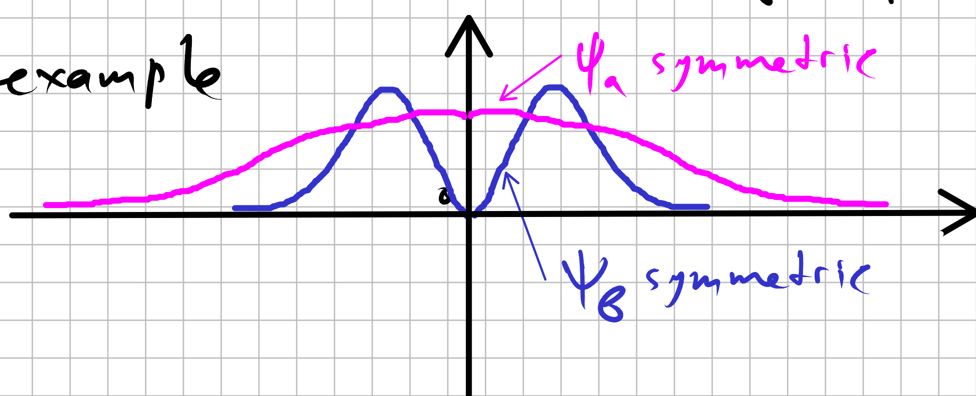
so bosons are closer and fermions are further than distinguishable particles, as if there is an exchange force. Techically it is not a force but a geometrical factor let's look at $\langle x \rangle_{ab} = \int x \, \underbrace{\psi_a^*(x) \, \psi_b(x)} \, dx$

for it to be non zero we need overlap between states 'a' and 'b'

So if particles far apart it is likely that they do not "feel" the exchange force

Of course being close together does not guarantee the exchange force existence

for example



$$\int x \, \psi_a(x) \, \psi_b(x) \, dx = 0$$