

Identical particles:

* Distinguishable

* Fermions

* Bosons

Suppose we have two identical particles,
i.e. we cannot tell them apart, they are indistinguishable

$$\Rightarrow \hat{H} = \hat{H}_{1,2}$$

If you blink and someone switches them,
the physical world is the same: $\hat{H}_{1,2} = \hat{H}_{2,1}$
also probability density must stay the same

$$|\Psi(\vec{r}_1, \vec{r}_2)|^2 = |\Psi(\vec{r}_2, \vec{r}_1)|^2$$

$$\Rightarrow \boxed{\Psi(\vec{r}_1, \vec{r}_2) = \pm \Psi(\vec{r}_2, \vec{r}_1)}$$

We can introduce **exchange operator**
 \hat{P} (do not confuse it with parity!)

$$\hat{P} \psi(\vec{r}_1, \vec{r}_2) = \psi(\vec{r}_2, \vec{r}_1)$$

Clearly $\hat{P} \hat{P} \psi(\vec{r}_1, \vec{r}_2) = \psi(\vec{r}_1, \vec{r}_2) \Rightarrow \hat{P} \lambda \psi(\vec{r}_2, \vec{r}_1) = \lambda^2 \psi(\vec{r}_1, \vec{r}_2)$

$$\lambda^2 = 1 \Rightarrow \lambda = \pm 1$$

$$\Rightarrow \hat{P} \psi(\vec{r}_1, \vec{r}_2) = \boxed{\psi(\vec{r}_2, \vec{r}_1) = \pm 1 \psi(\vec{r}_1, \vec{r}_2)}$$

same as above conclusion

even value of \hat{P}
symmetric wavefunction

anti-symmetric

One more property.

$$\boxed{[\hat{P}, \hat{M}] = 0}$$

Easy to prove if we recall
that $M_{1,2} = M_1 + M_2 = M_{2,1}$

Thus if state of the system is symmetric
or anti-symmetric, it will stay the same!

Particles which form symmetric states combination we call bosons,

and in anti-symmetric state — fermions

Note that we can have a compound particle

$$\psi(1, 2, 3, \dots, N) = +\psi(2, 1, 3, \dots) = +\psi(2, 3, 1, \dots) = \psi_+$$

or

$$= -\psi(2, 1, 3, \dots) = +\psi(2, 3, 1, \dots) = \psi_-$$

sign flips for pair wise exchange

boson

fermion

But because $[\hat{H}, \hat{p}] = 0$
once a boson — always a boson
fermion — — — fermion

By some "strange" reason bosons have integer spin
and fermions $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$

We need relativistic QM to prove it.

Note you can stick couple fermions together and make bosonic particle, if the total spin is integer

Let's try to construct a simplest
boson or fermion $\Psi_{\pm}(\vec{r}_1, \vec{r}_2)$

1st a little refresher: Ψ function of two non-interacting particles

$$\Psi(\vec{r}_1, \vec{r}_2), \quad \hat{H} = \underbrace{-\frac{\hbar^2}{2m_1} \nabla_1^2 + V_1(\vec{r}_1)}_{m_1} - \underbrace{\frac{\hbar^2}{2m_2} \nabla_2^2 + V_2(\vec{r}_2)}_{m_2}$$

We will seek for solution in the form $\Psi_a(\vec{r}_1) \cdot \Psi_b(\vec{r}_2)$

$$\hat{H} \Psi(\vec{r}_1, \vec{r}_2) = \hat{H} \Psi_a(\vec{r}_1) \cdot \Psi_b(\vec{r}_2) =$$

$$= (H_1 \Psi_a(\vec{r}_1)) \cdot \Psi_b(\vec{r}_2) + (H_2 \Psi_b(\vec{r}_2)) \cdot \Psi_a(\vec{r}_1) =$$

$$= E \Psi_a(\vec{r}_1) \cdot \Psi_b(\vec{r}_2) = (E_1 \Psi_a(\vec{r}_1)) \Psi_b + (E_2 \Psi_b(\vec{r}_2)) \Psi_a(\vec{r}_1)$$

\Rightarrow separation of variables

$$\left\{ \begin{array}{l} m_1 \Psi_a(\vec{r}_1) = E_1 \Psi_a(\vec{r}_1) \\ m_2 \Psi_b(\vec{r}_2) = E_2 \Psi_b(\vec{r}_2) \end{array} \right\} \begin{array}{l} 2 \text{ independent} \\ \text{Schrödinger equations} \end{array}$$

Important result: we just proved that the eigen state of two independent particles could be constructed as a product of two independent states

$$\Psi(\vec{r}_1, \vec{r}_2) = \Psi_a(\vec{r}_1) \cdot \Psi_b(\vec{r}_2)$$

Same true for N independent particles
 $\Psi(\vec{r}_1, \dots, \vec{r}_N) = \Psi_a(\vec{r}_1) \Psi_b(\vec{r}_2) \dots \Psi_z(\vec{r}_N)$

Now! the interesting part

Ψ_{\pm} it cannot be just $\Psi_a(r_1) \cdot \Psi_b(r_2)$
 since $\hat{P} \Psi_a(\vec{r}_1) \Psi_b(\vec{r}_2) \neq \pm \Psi_a(\vec{r}_2) \Psi_b(\vec{r}_1)$

but we can construct a linear combination

$$\Psi_{\pm}(\vec{r}_1, \vec{r}_2) = \frac{1}{\sqrt{2}} \left(\Psi_a(\vec{r}_1) \Psi_b(\vec{r}_2) \pm \Psi_a(\vec{r}_2) \Psi_b(\vec{r}_1) \right)$$

Check yourself that $\hat{P} \Psi_{\pm}(r) = \pm \Psi_{\pm}(r, 2)$

Note: \hat{P} switches either Ψ_a, Ψ_b or \vec{r}_1, \vec{r}_2 subindices
 not both!

We all heard many times that fermions cannot take the same state. Let's see why is it the case from exchange point of view

$$\text{Fermion} \Rightarrow \Psi_- = \frac{1}{\sqrt{2}} (\Psi_a(\vec{r}_1) \Psi_b(\vec{r}_2) - \Psi_a(\vec{r}_2) \Psi_b(\vec{r}_1))$$

now we put both particles to the same state 'a'

$$\Psi_-(\vec{r}_1, \vec{r}_2) = \frac{1}{\sqrt{2}} (\Psi_a(\vec{r}_1) \Psi_a(\vec{r}_2) - \Psi_a(\vec{r}_2) \Psi_a(\vec{r}_1))$$

if we apply exchange operator we expect

$$\hat{P} \Psi_-(\vec{r}_1, \vec{r}_2) = \Psi_-(\vec{r}_2, \vec{r}_1) = -\Psi_-(\vec{r}_1, \vec{r}_2)$$

$$\text{but in our case } \hat{P} \Psi_-(\vec{r}_1, \vec{r}_2) = \Psi_-(\vec{r}_2, \vec{r}_1) = +\Psi_-(\vec{r}_1, \vec{r}_2)$$

So we arrive to contradiction

our asymmetric wavefunction becomes symmetric!

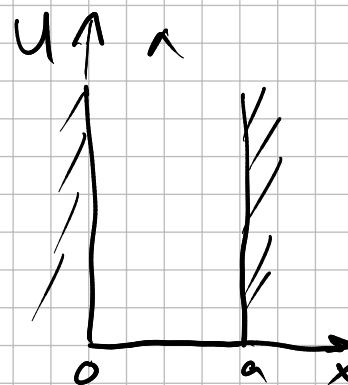
Wait a second, we all know that we can have 2 electron in for example 's' ($l=0$) orbital.

Yes, but they have different spin projection.

Their states are not exactly the same.

For a while we will forget about spin!
 We just care about is it fermions or bosons

let's put two non-interacting particles in the square well



For every particle
 $\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right)$, $E_n = n^2 E_1$

We have 3 cases

distinguishable particles

identical bosons

identical fermions

$$E_g = 2E_1$$

$$\psi_g = \psi_1(x_1) \cdot \psi_2(x_2) = \frac{2}{a} \sin\left(\frac{\pi}{a}x_1\right) \cdot \sin\left(\frac{2\pi}{a}x_2\right)$$

$$E_g = 2E_1$$

$$\psi_g = \frac{1}{\sqrt{2}} (\psi_1(x_1)\psi_1(x_2) + \psi_1(x_2)\psi_1(x_1))$$

There is no level !!!
 with $E_g = 2E_1$
 since it would 2 fermions in the same state

$$\psi_{fe} = \psi_1(x_1)\psi_2(x_2)$$

or... $\psi_2(x_1)\psi_1(x_2)$

degenerate
 $E_{fe} = 5E_1$

$$\psi_{fe} = \frac{1}{2} (\psi_1(x_1)\psi_2(x_2) + \psi_2(x_1)\psi_1(x_2))$$

non degenerate
 $E_{fe} = 5E_1$

The $E_g = 5E_1$

$$\psi_g(x_1, x_2) = \frac{1}{\sqrt{2}} (\psi_1(x_1)\psi_2(x_2) - \psi_2(x_1)\psi_1(x_2))$$

$$E_{fe} = (1+9)E_1, \psi_{fe} = \frac{1}{\sqrt{2}} (\psi_1(x_1)\psi_3(x_2) - \psi_3(x_1)\psi_1(x_2))$$