

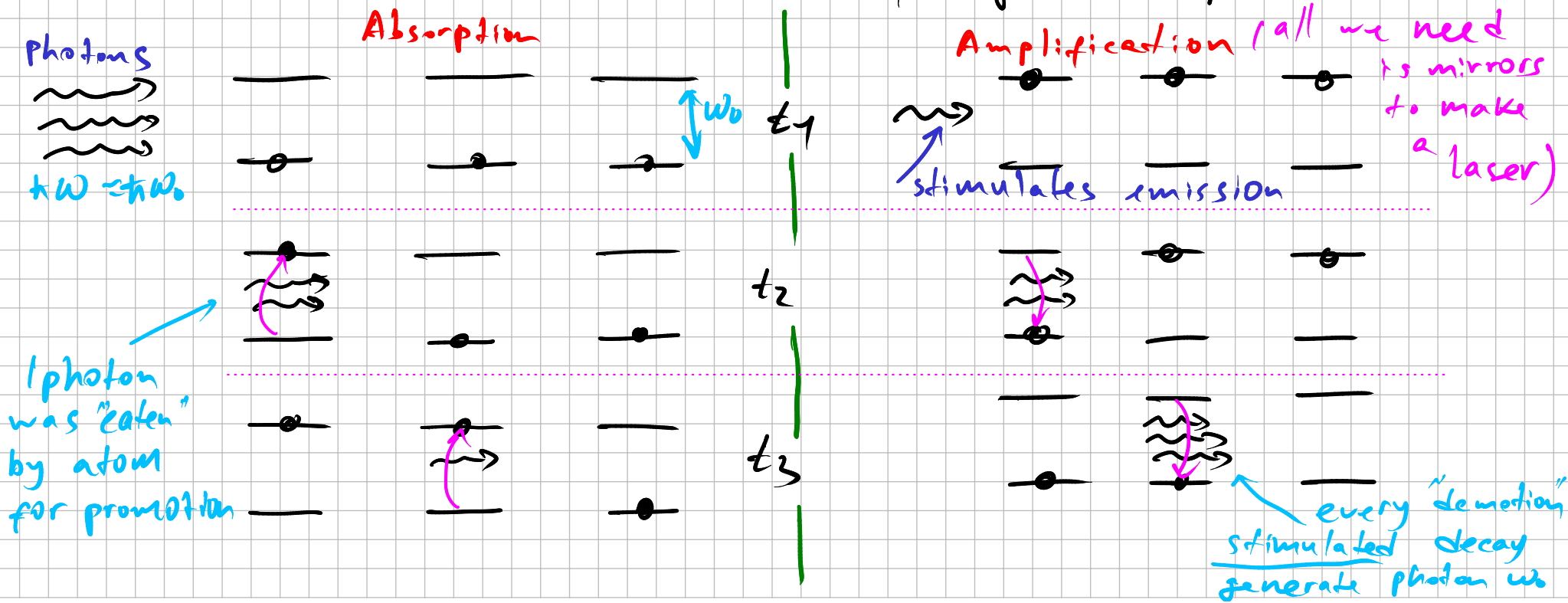
# Two-level atom spontaneous emission (decay)

$\beta$  ———  
 $\sim$  E<sub>osc</sub>(t)  
 $\alpha$  —●—

1st note that by swapping  $a \leftrightarrow b$  nothing changes in our equations

so if we found probability to "promote" (excite) from level  $a \rightarrow b$  it is the same as to decay from  $b \rightarrow a$  (if we start at 'b')

Now we can understand idea of light absorption



In either case we must have a changing  
in time Hamiltonian, otherwise  $c_n \psi_n$   
 $|c_n|^2$  becomes constant!

So QM states in general case  
 $\hat{H}$  independent of time  $\Rightarrow |c_n|^2 = \text{constant}$   $-iE_n t/\hbar$   
(remember about the phase term)

However experimentally we see that  
excited (higher energy) states decay down  
usually we blame external fields since  
there is no way to have an atom isolated.  
But actually the reason is more fundamental:  
there are vacuum fluctuations even if  
you deplete everything from excess energy  
make a light tight box there is  $\frac{1}{2}\hbar\omega$  field fluctuating  
ground energy of an oscillator

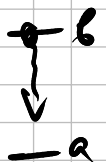
Quantization of light is subject of Quantum Electrodynamics (QED), so we just take a note that EM wave consist of photons  $\hbar\omega$  and can be treated as a harmonic oscillator.

The stronger (brighter) the light the more energetic it is i.e. more photons in it. But even pitch black situation carries  $\frac{1}{2}\hbar\omega$  vacuum fluctuations

So high energy decay is not spontaneous but stimulated by those vacuum fluctuations

We will attempt to estimate lifetime of an excited state somewhat simplistically.

1st recall probability of stimulated transition



$$P_{b \rightarrow a}(\omega) = \left( \frac{V_{ab}}{\hbar} \frac{\sin(\frac{\omega_0 - \omega}{2}t)}{\omega_0 - \omega} \right)^2$$

why we are not using exact case? B/c we have solution only for  $\omega_0 = \omega$  (resonant case)

recall  $V_{ab} = \langle a | q \hat{X} | b \rangle$

$= E \mathcal{P} \leftarrow$  dipole element, more precisely polarization thus curly  $\mathcal{P} \rightarrow \mathcal{P}$

Those vacuum fluctuations are incoherent  
 $\Rightarrow$  happen at all frequencies with no phase relationship

$$\text{so } P_{B \rightarrow a}(t) = \int_0^\infty d\omega \left( \frac{E(\omega) \rho}{\hbar} \cdot \frac{\sin \frac{\omega_0 - \omega}{2} t}{\omega_0 - \omega} \right)^2$$

For monochromatic field

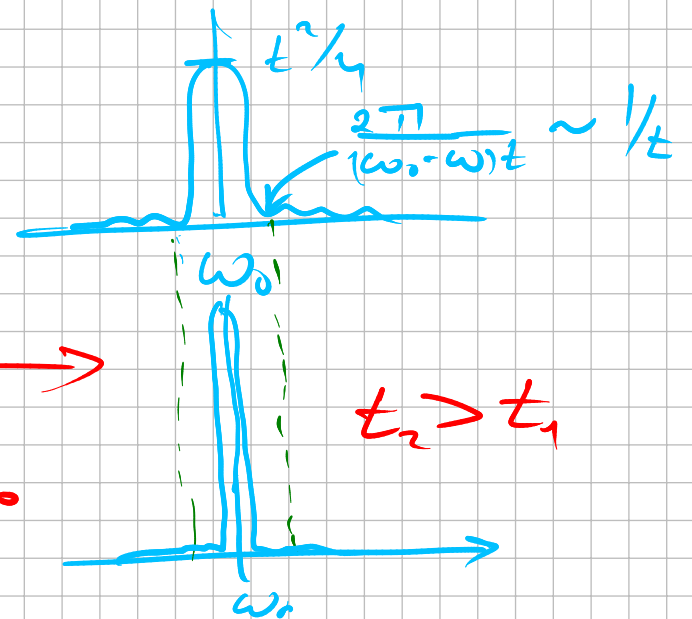
$$f(\omega) = \frac{\epsilon_0 E^2(\omega)}{2}$$

energy density of field  
 i.e. how much energy carried by given  $\omega$  field

$$P_{B \rightarrow a}(t) = \int \frac{2}{\epsilon_0} f(\omega) \left( \frac{\rho}{\hbar} \right)^2 \left( \frac{\sin \frac{(\omega_0 - \omega)}{2} t}{\omega_0 - \omega} \right)^2 d\omega$$

$$\left( \frac{\sin \frac{(\omega_0 - \omega)}{2} t}{(\omega_0 - \omega)/2} \right)^2$$

$$\xrightarrow{\omega_0 \rightarrow \omega} \frac{t^2}{4} \Rightarrow$$



$$\begin{aligned} & \parallel \\ & \frac{\pi}{2} \delta((\omega_0 - \omega)t) \\ & = \frac{\pi}{2} t \delta(\omega_0 - \omega) \end{aligned}$$

resembles  $\delta$  function for  $t \rightarrow \infty$

$$P_{b \rightarrow a}(t) = \frac{\pi}{\epsilon_0} \left| \frac{\mathcal{P}}{\hbar} \right|^2 g(\omega_0) t$$

We will be concerned with Black Body radiation which is everywhere and comes from all directions

$$\mathcal{P} = q \vec{r} \cdot \vec{n} \quad \begin{array}{l} \text{direction of} \\ \text{polarization} \\ \vec{r} \parallel \vec{z} \end{array}$$

$$= q z \cdot \cos \theta$$

$$\mathcal{P} = \int |\langle a | qz | b \rangle \cos \theta|^2 \cdot \sin \theta d\theta$$

$$= \frac{1}{3} |\langle a | qz | b \rangle|^2 = \frac{1}{3} \mathcal{P}_{ab}$$

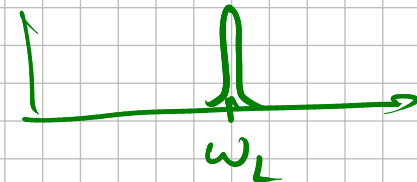
$$\Rightarrow \frac{d P_{b \rightarrow a}}{dt} =$$

$$\frac{\pi}{3 \epsilon_0} \left| \frac{\mathcal{P}_{ab}}{\hbar} \right|^2 g(\omega_0) = B_{b \rightarrow a} g(\omega_0)$$

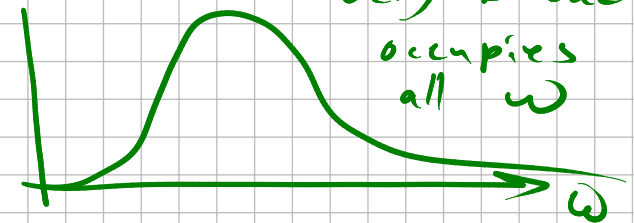
Some examples of  $g(\omega)$

laser (very close to monochromatic)

$$g(\omega) = \frac{\beta_0 \gamma^2}{(\omega - \omega_L)^2 + \delta^2}$$



Black body radiation very broad occupies all  $\omega$



Rate of transition in a unit of time

Now imagine many-many atoms  
 some are in state 'a' ( $N_a$  of them)  
 the others in state 'b' ( $N_b$ )

in a given time some of them spontaneously  
 decay with rate 'A'  $\Rightarrow \frac{dN_b}{dt} = -A \cdot N_b$

—b

some will be stimulated down

with  $-B_{ba} \cdot \rho(\omega_0) N_b$  some promoted up

with  $+B_{ab} \rho(\omega_0) N_a$

more atoms in state b, more decays we see

Einstein coefficient

$$\Rightarrow \frac{dN_b}{dt} = -A N_b - B_{ba} \rho(\omega_0) N_b + B_{ab} \rho(\omega_0) N_a \quad (*)$$

Black Body radiation is constant in equilibrium

thus number of decays  $\frac{dN_b}{dt} = 0$  otherwise we amplify the field

$$\Rightarrow \rho(\omega_0) = \frac{A \cdot N_b}{B_{ab} N_a - B_{ba} N_b} = \frac{A}{\frac{N_a}{N_b} B_{ab} - B_{ba}}$$

Stat Mechanics says Boltzmann distribution

$$\frac{N_a}{N_b} = \frac{e^{-E_a/kT}}{e^{-E_b/kT}} = e^{h\omega_0/kT}$$

$$f(\omega_0) = \frac{A}{e^{\frac{\hbar\omega_0}{kT}} B_{ab} - B_{ba}}$$

We can compare it with Black Body radiation density (spectrum)

$$f_{BB}(\omega) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{e^{\frac{\hbar\omega}{kT}} - 1} = f(\omega_0)$$

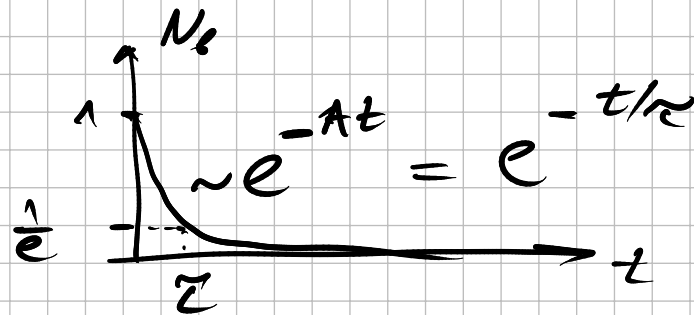
$$\Rightarrow B_{ab} = B_{ba} \text{ and } = \frac{\pi}{3\epsilon_0} \left| \frac{p_{ab}}{\hbar} \right|^2 \text{ we already know it}$$

$$\frac{A}{B_{ab}} = \frac{\hbar \omega_0^3}{\pi^2 c^3} \Rightarrow \boxed{A = \frac{\hbar \omega_0^3}{3\pi c^3 \epsilon_0} |p_{ab}|^2}$$

What is the meaning of 'A'.

Let's look at (x) and think about single atom: once photon is emitted there is nothing to stimulate and there are no photons from outside  $\Rightarrow \rho(\omega_s) = 0$

$$\Rightarrow \frac{dN_0}{dt} = -A N_0$$
$$N_0 = \underbrace{N_0(0)}_1 e^{-A \cdot t}$$



so life time of excited state

$$\tau = 1/A$$

recall that it  $\neq 0$  for  $\Delta l = 1$   
 $\Delta m = \pm 1, 0$

Note  $A \sim |\rho_{ab}|^2 = |\rho_{ba}|^2$

the stronger the matrix element

the stronger the transition

i.e. chance of spontaneous decay  
but thus of stimulated transition

Optics  
Jargon