Two level atom – with pereodic exitation

"There are no two level atoms and Rubidium is not one of them"

Bill Philips (Noble prize winner)

Nevertheless it is useful aproximation

$$|b\rangle \Rightarrow E_b$$

$$E_b - E_a = \hbar\omega_0$$

$$|a\rangle \Rightarrow E_a$$

$E\cos(\omega t)$

EM wave

$$H_1 \sim E q \hat{x}$$

$$H_{1\,aa} = \langle a|\, E q \hat{x}\, |a\rangle = 0$$

symmetry

$$H_{1\,bb} = 0$$

$$H_{1\,ab} = \langle a|\, E q \hat{x}\, |b\rangle = V_{ab} \cos(\omega t)$$

Last time we derived

$$c_f(t) = \delta_{fi} - \frac{i}{\hbar}\int_0^t e^{i(E_f - E_i)t'/\hbar} \langle f|\, H_1\, |i\rangle \, dt$$

$|c_a(0)|^2 = 1$ i.e. atoms initially in 'a' state

$|c_b(0)|^2 = 0$

$$c_b(t) = -\frac{i}{\hbar}\int_0^t e^{i\omega_0 t'} V_{ab}\cos(\omega t')\,dt'$$

$$= -\frac{i}{\hbar}V_{ab}\int_0^t e^{i\omega_0 t'}\frac{\left(e^{i\omega t'} + e^{-i\omega t'}\right)}{2}\,dt'$$

$$= -\frac{iV_{ab}}{2\hbar}\left[\frac{e^{i(\omega_0+\omega)t'}}{\omega_0+\omega} + \frac{e^{i(\omega_0-\omega)}}{\omega_0-\omega}\right]\Bigg|_0^{t'}$$

$$= -\frac{iV_{ab}}{2\hbar}\left[\cancel{\frac{e^{i(\omega_0+\omega)t}-1}{\omega_0+\omega}} + \frac{e^{i(\omega_0-\omega)t}-1}{\omega_0-\omega}\right]$$

0 $<<$

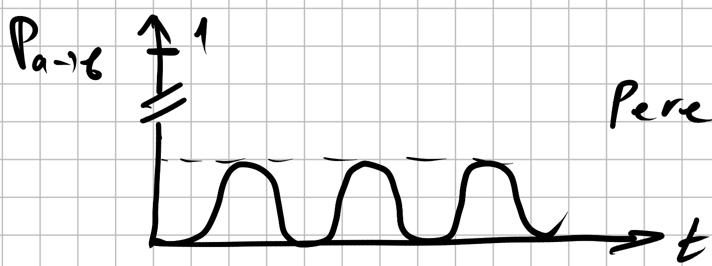
for optical transitions $\omega_0 \simeq 10^{14}\,Hz$, RF $\omega_0 \simeq 10^{10}\,Hz$

$$c_b(t) = -\frac{iV_{ab}}{2\hbar}\,\frac{e^{i(\omega_0-\omega)t}-1}{\omega_0-\omega} = -\frac{iV_{ab}}{\hbar}\,(i)\,\frac{e^{i\frac{(\omega_0-\omega)}{2}t} - e^{-i\frac{(\omega_0-\omega)}{2}t}}{i\,2\,(\omega_0-\omega)}\,e^{+i\frac{(\omega_0-\omega)}{2}t}$$

$$c_b(t) = \frac{V_{ab}}{\hbar} \frac{\sin\left(\frac{\omega_0-\omega}{2}t\right)}{\omega_0-\omega} e^{i\frac{(\omega_0-\omega)}{2}t}$$

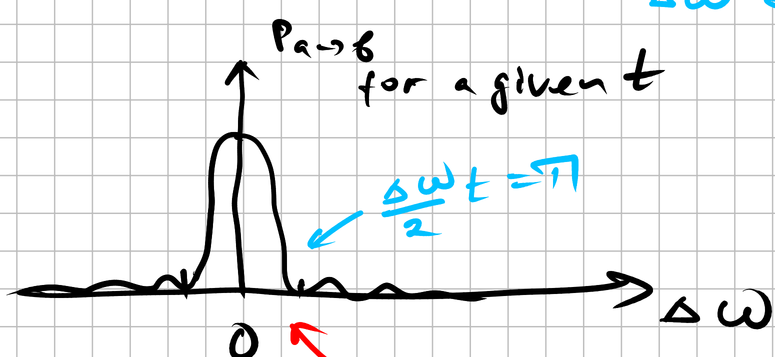
$$\Rightarrow \boxed{P_{a\to b}(t) = |c_b(t)|^2 = \frac{|V_{ab}|^2}{\hbar^2} \frac{\sin^2\left(\frac{\omega_0-\omega}{2}t\right)}{(\omega_0-\omega)^2}}$$

probability to move from level 'a' to 'b'



Periodic oscillations (flopping)

$$P_{a\to b\,max} = \left|\frac{V_{ab}}{\hbar\underbrace{(\omega_0-\omega)}_{\Delta\omega\ \text{detuning}}}\right|^2 = \left|\frac{V_{ab}}{\hbar\Delta\omega}\right|^2 \quad \text{at} \quad \frac{\omega_0-\omega}{2}t = \frac{\pi}{2}+n\pi$$



narrow resonance

one issue

$P_{a\to b\,max} \xrightarrow[\Delta\omega\to 0]{} \infty$

1st it cannot be >1

$\Rightarrow$ 2nd we out of perturbation regime!

Two level atom exact (almost) solution

Recall that during time perturbation derivation we got the exact expressions for $C_n(t)$

$$\frac{d}{dt} C_f(t) = -\frac{i}{\hbar} \sum_n C_n(t)\, e^{\frac{i(E_f^{(0)} - E_n^{(0)})t}{\hbar}} \langle \psi_f | H_1(t) | \psi_n \rangle$$

for the two level case and H_1 which has only off diagonal elements

$$H_{1ab} = V_{ab} \cos(\omega t)$$

we have

$$\begin{cases} \frac{d}{dt} C_a(t) = -\frac{i}{\hbar} C_b(t)\, e^{-i\omega_0} V_{ab} \cos(\omega t) \\ \frac{d}{dt} C_b(t) = -\frac{i}{\hbar} C_a(t)\, e^{+i\omega_0} V_{ab}^{*} \cos(\omega t) \end{cases}$$

$$
\begin{cases}
\dot{c}_a(t) = -\frac{i}{\hbar} c_b(t)\, e^{-i\omega_0} \dfrac{e^{i\omega t} + \cancel{e^{-i\omega t}}}{2} V_{ab} \\
\dot{c}_b(t) = -\frac{i}{\hbar} c_a(t)\, e^{+i\omega_0} \dfrac{e^{-i\omega t} + \cancel{e^{i\omega t}}}{2} V_{ab}^*
\end{cases}
$$

$e^{-i(\omega_0+\omega)t}$ fast oscillations

$e^{+i(\omega_0+\omega)t}$ we drop them

$H_1 = V_{ab}\cos(\omega t) \Rightarrow V_{ab}\, e^{-i\omega t}$

Rotating wave approximation

Why do we drop them?

they would force c_a, c_b oscillate with frequency $(\omega_0+\omega) \sim 10^{14}$ which is HUGE

there are no detectors which are that fast!

So, who cares about such terms.

$$
\begin{cases}
\dot{c}_a(t) = -\frac{i}{2\hbar} c_b(t)\, e^{-i(\omega_0-\omega)t} V_{ab} \\
\dot{c}_b(t) = -\frac{i}{2\hbar} c_a(t)\, e^{i(\omega_0-\omega)t} V_{ab}^*
\end{cases}
$$

There is a general solution but we will do the case of zero detuning $\Delta\omega = 0 \iff \omega_0 = \omega$

$$\begin{cases} \dot{C}_a = -\frac{i}{2\hbar} C_b V_{ab} \\ \dot{C}_b = -\frac{i}{2\hbar} C_a V_{ba} \end{cases}$$

$$\begin{cases} \ddot{C}_a = -\frac{i}{2\hbar} \dot{C}_b V_{ab} = -\frac{V_{ab} \cdot V_{ba}}{\hbar^2} C_a = -\frac{|V_{ab}|^2}{\hbar^2} C_a \\ \ddot{C}_b = -\frac{|V_{ab}|^2}{4\hbar^2} C_b \end{cases}$$

Well we know the solution $C = A \cos \Omega t + B \sin \Omega t$

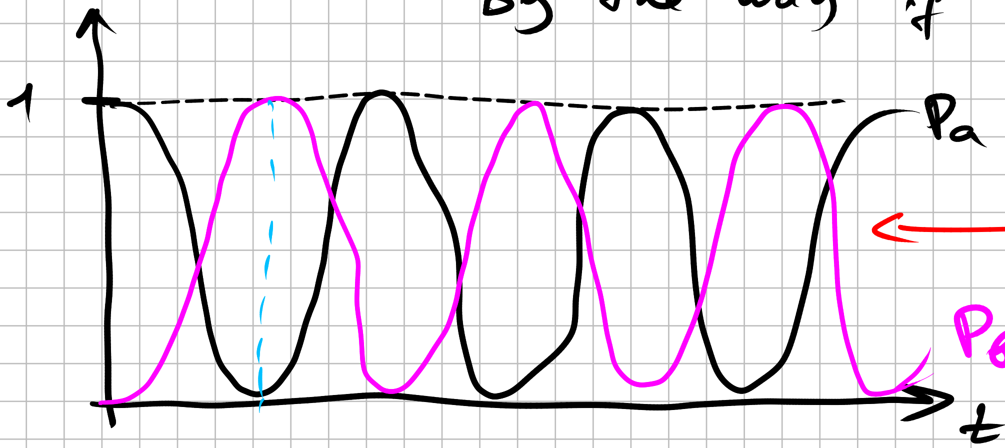
where $\boxed{\Omega = \frac{|V_{ab}|}{2\hbar}, \text{ Rabi frequency}}$

Let's use initial condition $C_a(0) = 1$ ← everything at ground level

$C_b(0) = 0$

$C_a = \cos(\Omega t)$, $C_b = \sin(\Omega t)$

by the way if $\Delta\omega = 0 \Rightarrow$ $\boxed{\Omega = \frac{1}{2}\sqrt{\Delta\omega^2 + \left(\frac{|V_{ab}|}{\hbar}\right)^2}}$



$P_a = |C_a|^2$

Rabi flopping

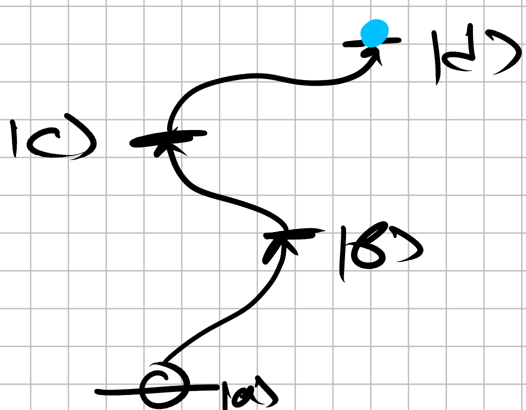
$P_b = |C_b|^2$

τ : $\Omega\tau = \pi$

so called π pulse time

which brings all population from 'a' to 'b'

Now we can do quantum control



i.e. populate any level even if we cannot generate $\omega = \omega_d - \omega_a$

HW: show that for $\Delta\omega = 0$ and $t \to 0$ perturbation theory gives the same answer