

Time independent perturbation with degenerate levels.

Note that we generally need a terms

Like
$$\frac{\langle \Psi_k^{(0)} | H_1 | \Psi_n^{(0)} \rangle}{E_n^{(0)} - E_k^{(0)}}$$

but if we have degeneracy $E_n = E_k$ the above term explodes, since we have $\frac{1}{0}$

\Rightarrow we need different strategy to handle perturbation of degenerate levels

Side note: we cannot even do 1st order correction

$$E_n^{(1)} = \langle \Psi_n | H_1 | \Psi_n \rangle$$

Since if value $\Psi_a^{(0)}, \Psi_b^{(0)}, \Psi_c^{(0)}$ are eigenstates of \hat{H}_0 with the same eigenvalue $E_n^{(0)}$ than any linear combination is eigenstate too
 $|\Psi_n^{(0)}\rangle = C_a |\Psi_a^{(0)}\rangle + C_b |\Psi_b^{(0)}\rangle + C_c |\Psi_c^{(0)}\rangle + \dots$
but different set of 'C' form different $|\Psi_n^{(0)}\rangle$

But we can reuse old approach with small modifications. We are solving:

$$(\hat{H}_0 + \lambda \hat{H}_1) \psi_n = E_n \psi_n =$$

as before $E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \dots$

$$\psi_n = \psi_n^{(0)} + \lambda \psi_n^{(1)} + \lambda^2 \psi_n^{(2)} + \dots$$

Focusing only on terms $\sim \lambda$ we have

$$\hat{H}_0 |\psi_n^{(1)}\rangle + \hat{H}_1 |\psi_n^{(0)}\rangle = E_n^{(0)} |\psi_n^{(1)}\rangle + E_n^{(1)} |\psi_n^{(0)}\rangle \quad *$$

so far no surprises we saw it before

Here is the difference, 'n' state is degenerate

$$\Rightarrow \psi_n^{(0)} = c_a |\psi_a^{(0)}\rangle + c_b |\psi_b^{(0)}\rangle + c_c |\psi_c^{(0)}\rangle + \dots = c_i |\psi_i^{(0)}\rangle$$

we drop 'n' subindex, otherwise notation is too bulky. But we must remember about $\hat{H}_0 |\psi_{a,b,c,\dots}\rangle = E_n |\psi_{a,b,c,\dots}\rangle$

let's calculate $\langle \psi_a^{(0)} | * \rangle$

$$\langle \psi_a^{(0)} | \hat{H}_0 | \psi_n^{(1)} \rangle + \langle \psi_a^{(0)} | \hat{H}_1 | \psi_n^{(0)} \rangle = \langle \psi_a^{(0)} | E_n^{(0)} | \psi_n^{(1)} \rangle + \langle \psi_a^{(0)} | E_n^{(1)} | \psi_n^{(0)} \rangle$$

$\ll \langle \psi_a^{(0)} | \hat{H}_0 = E_n^{(0)} \langle \psi_a^{(0)} |$

$$\langle \psi_a | H_1 | \Psi_n^{(1)} \rangle = E_n^{(1)} \langle \psi_a | \Psi_n^{(1)} \rangle = E_n^{(1)} \cdot c_a$$

recall $\Psi_n^{(1)} = c_a |\psi_a\rangle + c_b |\psi_b\rangle + c_c |\psi_c\rangle + \dots$

$$c_a \underbrace{\langle \psi_a | H_1 | \psi_a \rangle}_{H_{1aa}} + c_b \underbrace{\langle \psi_a | H_1 | \psi_b \rangle}_{H_{1ab}} + c_c \underbrace{\langle \psi_a | H_1 | \psi_c \rangle}_{H_{1ac}} + \dots = E_n^{(1)} \cdot c_a$$

We can generalize

$$\langle \psi_m | \otimes \rangle \Rightarrow c_a H_{1ma} + c_b H_{1mb} + c_c H_{1mc} + \dots = E_n^{(1)} c_m$$

or in matrix notation

$$\begin{pmatrix} H_{1aa} & H_{1ab} & H_{1ac} & \dots \\ H_{1ba} & H_{1bb} & H_{1bc} & \dots \\ H_{1ca} & H_{1cb} & H_{1cc} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} c_a \\ c_b \\ c_c \\ \vdots \end{pmatrix} = E_n^{(1)} \begin{pmatrix} c_a \\ c_b \\ c_c \\ \vdots \end{pmatrix}$$

$$\tilde{H}_1 \cdot \mathbf{c} = E_n^{(1)} \cdot \mathbf{c}$$

\mathbf{c} is eigenvector
 $E_n^{(1)}$ is eigenvalue

\hat{H}_1 in basis of n^{th} level degenerate states

Slightly rearranging above equation

$$\begin{pmatrix} H_{1aa} - E_n^{(1)} & H_{1ab} & H_{1ac} & \dots & \dots \\ H_{1ba} & H_{1bb} - E_n^{(1)} & H_{1bc} & \dots & \dots \\ H_{1ca} & H_{1cb} & H_{1cc} - E_n^{(1)} & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} c_a \\ c_b \\ c_c \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \end{pmatrix}$$

$$(\tilde{H}_1 - E_n^{(1)} \tilde{I})$$

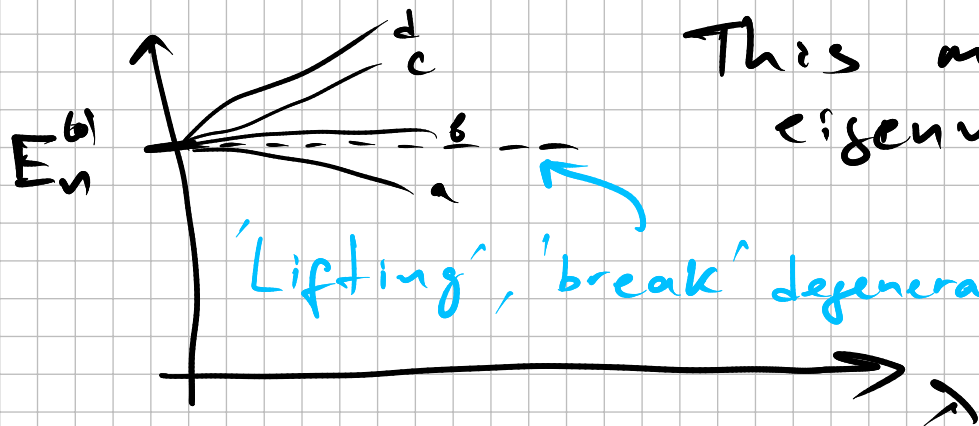
identity = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

characteristic equation

to find $E_n^{(1)}$ we solve

$$\det(\tilde{H}_1 - E_n^{(1)} \tilde{I}) = 0$$

in general there are as many solutions as is degeneracy of the 'n' level



This means several eigenvectors c_i as well!

If off diagonal terms of $\tilde{H}_1 = 0$,
 i.e. \tilde{H}_1 is diagonal then it is easy to
 find solution an 'C' combinations

$$\det \begin{pmatrix} H_{1aa} - E_n^{(1)} & 0 & 0 \\ 0 & H_{1bb} - E_n^{(1)} & 0 \\ 0 & 0 & H_{1cc} - E_n^{(1)} \end{pmatrix} = 0$$

$$(H_{1aa} - E_n^{(1)}) (H_{1bb} - E_n^{(1)}) (H_{1cc} - E_n^{(1)}) \dots = 0$$

$$\Rightarrow E_n^{(1)} = H_{1aa}, H_{1bb}, H_{1cc}, \dots$$

$$C_{\downarrow} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \dots$$

There are gazillion ways to express any of constituting degenerate states, e.g. ψ_a

it would be lovely to "guess" a set of $\psi_a, \psi_b, \psi_c \dots$ so \tilde{M}_1 is diagonal.

Unfortunately, there is no constructive way to do it, before solving "characteristic" equation.

We can search for states of operator \hat{O} which commute with \hat{H}_1 and \hat{H}_0 ,

then in $|0\rangle$ basis our \tilde{M}_1 matrix will be diagonal. But this shifts the question to "how to find such \hat{O} ?" which in general not easy.