Time dependent perturbations

So far we were focusing on time independent Schrödinger equation. $\hat{H} \neq \hat{H}(t)$

$$i\hbar \frac{d\psi}{dt} = \hat{H}\psi$$

which we obtain by doing variable separation $\psi(\vec{r},t) = e^{-\frac{iEt}{\hbar}} \psi(\vec{r})$

$\Rightarrow \hat{H}\psi(\vec{r}) = E\psi(\vec{r})$

So general solution is

$$\psi(\vec{r},t) = \sum_n e^{-\frac{iE_n t}{\hbar}} c_n \cdot \psi_n(\vec{r}) \qquad (*)$$

$\vec{r}$ could be anyting $(n, \ell, m \dots)$ but time

boring phase factor

Now we add time dependence and split Hamiltonian to $\hat{H}_0 + \hat{H}_1(t)$

let's assume we can solve

$\hat{H}_0 \psi_n^{(0)} = E_n^{(0)} \psi_n^{(0)}$, then $\{\psi_n^{(0)}\}$ is the full set and any function can be expessed as $\sum_n c_n \psi_n^{(0)}$

this is true for $\psi(t,\vec{r})$: $\hat{H}\,\psi(t,\vec{r}) = (\hat{H}_0 + \hat{H}_1(t))\,\psi(t,\vec{r}) = i\hbar \frac{d}{dt}\,\psi(t,\vec{r})$

$$\psi(t,\vec{r}) = \sum_n c_n(t)\, e^{-iE_n^{(0)}t/\hbar}\, \psi_n^{(0)}(\vec{r})$$

comparing with (*) c_n is time dependent!

for convinience we drop $\vec{r}$: $\psi(t,\vec{r}) \longrightarrow \psi(t)$, $\psi^{(0)}(\vec{r}) \to \psi^{(0)}$

$$\Rightarrow (\hat{H}_0 + \hat{H}_1)|\psi(t)\rangle = \sum c_n(t)\, e^{-\frac{iE_n^{(0)}t}{\hbar}} (\hat{H}_0 + \hat{H}_1)|\psi_n^{(0)}\rangle =$$

$E_n^{(0)}$

$$= i\hbar \frac{d}{dt}|\psi(t)\rangle = i\hbar \sum_n \left(\frac{d}{dt} c_n(t) - \cancel{\frac{iE_n^{(0)}}{\hbar} c_n(t)} \right) e^{-iE_n^{(0)}t/\hbar} |\psi_n^{(0)}\rangle$$

$$= \sum_n c_n(t)\, e^{-iE_n^{(0)}t/\hbar} (\cancel{E_n^{(0)}} + H_1)|\psi_n^{(0)}\rangle$$

let's "multiply" by $\langle \psi_f^{(0)}|$ and recall $\langle \psi_f^{(0)} | \psi_n^{(0)} \rangle = \delta_{fn}$

$$\boxed{\frac{d}{dt} c_f(t) = \frac{-i}{\hbar} \sum_n c_n(t)\, e^{\frac{i(E_f^{(0)} - E_n^{(0)})t}{\hbar}} \langle \psi_f^{(0)} | H_1(t) | \psi_n^{(0)} \rangle}$$

this exact but complicated
since $\sum_n c_n$, contains c_f

We will do perturbation trick
and assume $\hat{H}_1$ small $\Rightarrow$ $\boxed{\hat{H}_1 \to \lambda \hat{H}_1}$

$$c_n(t) = c_n^{(0)} + \lambda c_n^{(1)} + \lambda^2 c_n^{(2)} \ldots \quad , \quad \lambda \ll 1$$

Collecting terms $\sim \lambda^0$ we see

$$\boxed{\frac{d}{dt} c_f^{(0)} = 0} \quad , \quad \text{since } \langle \psi_f^{(0)} | \lambda \hat{H}_1 | \psi_n^{(0)} \rangle \sim \lambda \ldots$$

Next we assume that at initial time (usually 0 or $-\infty$)
we will use 0 here
the system "sits" at energy level 'i'
and then $H_1(t)$ is "on"

$$\Rightarrow \quad c_n^{(0)} = \delta_{ni} \quad \text{and} \quad c_n^{(k)}(0) = 0 \text{ , for any } k \geq 1$$

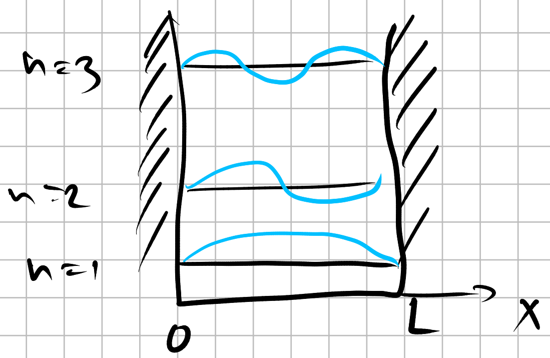
let's focus on λ^1 term

$$\frac{d}{dt} c_f^{(1)}(t) = -\frac{i}{\hbar} \sum_n \underbrace{c_n^{(0)}(t)}_{\delta_{ni}} e^{i(E_f - E_n)t/\hbar} \langle \psi_f^{(0)} | H_1 | \psi_n^{(0)} \rangle$$

$$= -\frac{i}{\hbar} e^{i(E_f - E_n)t/\hbar} \langle \psi_f^{(0)} | H_1 | \psi_i^{(0)} \rangle$$

$$\Rightarrow \quad \boxed{c_f(t) = \overset{c_f^{(0)}}{\delta_{fi}} - \underbrace{\frac{i}{\hbar}\int_0^t e^{i(E_f - E_i)t'/\hbar} \langle \psi_f^{(0)} | \hat{H}_1(t) | \psi_i^{(0)} \rangle \, dt}_{c_f^{(1)}\text{, here we did } \lambda H_1 \to H_1 \text{ "back conversion"}}}$$

Example:

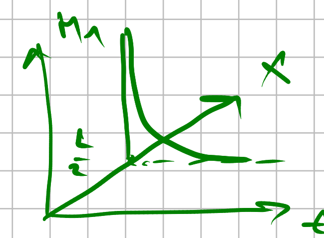


$$\psi_n^{(0)} = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi}{L} n x\right)$$

$$H_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$$

$$V(x) = \begin{cases} 0, & 0 \le x \le L \\ \infty, & \text{elsewhere} \end{cases}$$

$$H_1(t) = \alpha\, \delta\left(x - \frac{L}{2}\right) e^{-t/\tau}$$



let's set initial state to be $n = 1$

kronecker, do not confuse with

$$c_f(t) = \delta_{f1} - \frac{i}{\hbar}\int_0^t dt' e^{i(E_f - E_i)t'/\hbar} \int_0^L dx \frac{2}{L} \sin\left(\frac{\pi}{L} f x\right) \cdot \sin\left(\frac{\pi}{L} x\right) \alpha\, \delta\left(x - \frac{L}{2}\right) e^{-t'/\tau}$$

$$= \delta_{f1} - \frac{i}{\hbar}\int_0^t dt' e^{i(E_f - E_i)t'/\hbar} \frac{2\alpha}{L} \underbrace{\sin\left(\frac{\pi}{2} f\right) \sin\left(\frac{\pi}{2}\right)}_{\neq 0 \text{ only for odd } f} e^{-t'/\tau}$$

$$= \delta_{f1} - \frac{i\, 2\alpha}{L} \sin\left(\frac{\pi}{2} f\right) \left[\frac{e^{-\frac{t}{\tau} + i(E_f - E_i)t/\hbar} - 1}{-\frac{1}{\tau} + i\frac{(E_f - E_i)}{\hbar}} \right]$$

$$(E_f - E_i)/\hbar = \omega_{fi} \quad \text{oscillation frequency}$$

$$c_f(t) = \delta_{f1} - \frac{i2\alpha}{L}\sin\left(\frac{\pi}{2}f\right)\frac{e^{-t/\tau + i\omega_{fi}t} - 1}{-\frac{1}{\tau} + i\omega_{fi}}$$

Probability to be detected in state $f = 2m+1 \neq 1$

$$\underset{\overset{\shortparallel}{f}}{P_{2m+1}} = \left|\frac{2\alpha}{L}\right|^2 \frac{\left(e^{-t/\tau}\cdot e^{i\omega_{f1}t} - 1\right)\left(e^{-t/\tau}e^{-i\omega_{f1}t} - 1\right)}{\left(\frac{1}{\tau}\right)^2 + \omega_{f1}^2}$$

$$\left(e^{-t/\tau}e^{i\omega_{f1}t} - 1\right)\left(e^{-t/\tau}e^{-i\omega_{f1}t} - 1\right) =$$

$$= e^{-2t/\tau} + 1 - e^{-t/\tau}\big(\underbrace{e^{i\omega_{f1}t} + e^{-i\omega_{f1}t}}_{2\cos\omega_{f1}t}\big)$$

$$P_{f=2m+1} = \left|\frac{2\alpha\tau}{L}\right|^2 \frac{1 + e^{-\frac{2t}{\tau}} - 2\cos(\omega_{f1}t)e^{-t/\tau}}{1 + (\omega_{f1}\tau)^2}$$

for very long τ, and $t \ll \tau$,

$$P_{f=2m+1} = \left| \frac{2\alpha\tau}{L} \right|^2 \frac{4\sin^2\left(\frac{\omega_{f1} t}{2}\right)}{1 + (\omega_{f1}\tau)^2}$$

$t \gg \tau$

$$P_{f=2m+1} = \left| \frac{2\alpha\tau}{L} \right|^2 \frac{1}{1 + (\omega_{f1}\tau)^2}$$