Scattering

The Born approximation

– Integral form of solution for Shrödinger equation

$$(\nabla^2\psi + k^2\psi)\psi = \frac{2m}{\hbar^2} V\psi = Q \quad , \; k^2 = \frac{2mE}{\hbar^2}$$

(*) $$\boxed{\psi(\vec r) = \psi_0(\vec r) - \frac{2m}{\hbar^2}\int \frac{e^{ik|\vec r - \vec r'|}}{4\pi|\vec r - \vec r'|} V(\vec r')\psi(\vec r')\, d^3\vec r'}$$

, where ψ_0 satisfies source free equation $(\nabla^2 + k^2)\psi_0(\vec r) = 0$

Let's focus on solution in far zone $r \to \infty$

Assuming $V(\vec r') \xrightarrow[r' \to \infty]{} 0$ i.e. we interested in case $r' \ll r$

we approximate

$$|\vec r - \vec r'| = \sqrt{(\vec r - \vec r')^2} = \sqrt{r^2 - 2\vec r \vec r' + \cancelto{0}{r'^2}}$$

$$\approx \sqrt{r^2\left(1 - \frac{2\vec r \cdot \vec r'}{r^2}\right)} = r\sqrt{1 - \underbrace{\frac{2\hat r \cdot \vec r'}{r}}_{\ll 1}}$$

$$\approx r - \hat r \vec r'$$

unit vector $\hat r = \frac{\vec r}{r}$

Let's define $\vec{k} = k \cdot \hat{r} \longrightarrow (\ast)$

we keep this
it is oscillating term

$$\psi(\vec{r}) = \psi_0(\vec{r}) - \frac{m}{2\pi\hbar^2} \int \frac{e^{ikr - i\vec{k}\vec{r}'}}{|\vec{r} - \cancel{\hat{r}\vec{r}'}|} V(\vec{r}')\,\psi(\vec{r}')\,d^3\vec{r}'$$

0 we neglect small correction

$$\psi(\vec{r}) = \psi_0(\vec{r}) - \frac{m}{2\pi\hbar^2} \frac{e^{ikr}}{r} \int e^{-i\vec{k}\vec{r}'} V(\vec{r}')\,\psi(\vec{r}')\,d^3\vec{r}'$$

recall $\psi(\vec{r}) = A\left(e^{ikz} + f(\theta)\,\frac{e^{ikr}}{r}\right)$

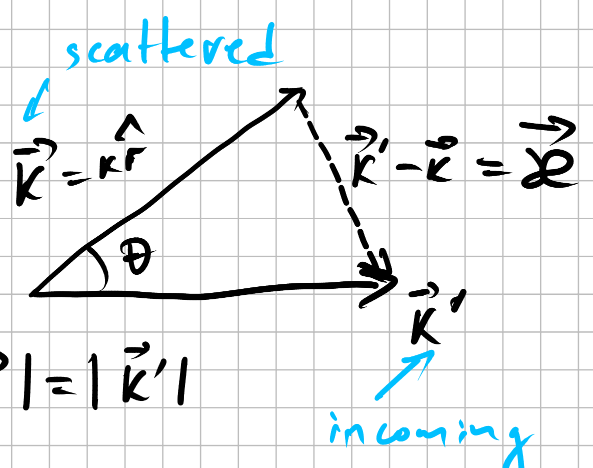
$$\Rightarrow f(\theta) = -\frac{m}{2\pi\hbar^2}\frac{1}{A} \int e^{-i\vec{k}\vec{r}'} V(\vec{r}')\,\psi(\vec{r}')\,d^3\vec{r}'$$

Born approximation $\psi(\vec{r}) \simeq \psi_0(\vec{r}) = A e^{ikz} = A e^{i\vec{k}'\vec{r}'}$

weak potential

almost no change to incoming wave

$\vec{k}' = k\hat{z}$



$|\vec{k}| = |\vec{k}'|$

3D Fourier transform

$$\boxed{f(\theta) = -\frac{m}{2\pi\hbar^2} \int e^{i\underbrace{(\vec{k}' - \vec{k})}_{\varkappa}\vec{r}'} V(\vec{r}')\,d^3\vec{r}'}$$

Spherically symmetric potential

$$V(\vec r) = V(r)$$

$$f(\theta) = -\frac{m}{2\pi\hbar^2}\int e^{i\vec\varkappa\vec r'}\, V(r')\, r'^2 \sin(\theta')\, d\theta'\, \overbrace{d\varphi'}^{2\pi}\, dr'$$

$$\vec\varkappa\vec r' = \varkappa r' \cos\theta'$$ (we direct z' along $\vec\varkappa$)

$$f(\theta) = -\frac{m}{2\pi\hbar^2}\,2\pi\int e^{i\varkappa r'\cos\theta'}\, r'^2 dr \cdot d(-\underbrace{\cos\theta'}_{t})\, V(r')$$

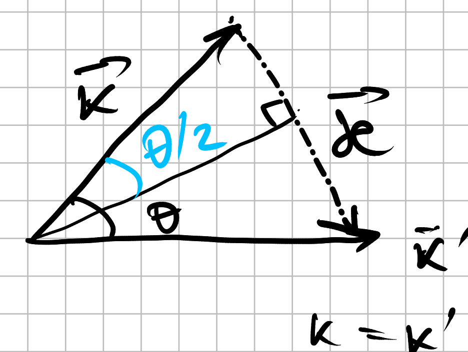
$$= -\frac{m}{2\pi\hbar^2}\int r'^2 dr \left(-\frac{e^{i\varkappa r' t}}{i\varkappa r'}\Bigg|_{1}^{-1}\right) V(r') =$$

$$= \frac{m}{\hbar^2}\int \frac{r' dr'}{i\varkappa}\underbrace{\left(e^{-i\varkappa r'} - e^{i\varkappa r'}\right)}_{-2i\sin(\varkappa r')} V(r') =$$

$$\boxed{f(\theta) = -\frac{2m}{\hbar^2}\,\frac{1}{\varkappa}\int_0^\infty \sin(\varkappa r')\, r'\, V(r')\, dr'}$$

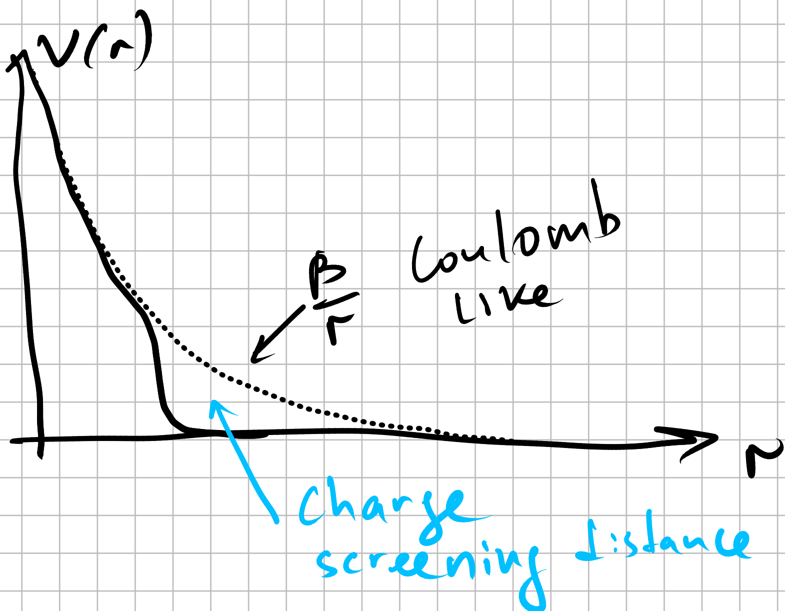
Where is θ dependence in ↑ ?

$$\boxed{\varkappa = 2k\sin(\theta/2)}$$



Example: Yukawa potential

$$V(r) = \beta \frac{e^{-\mu r}}{r}$$



$r' \to r$

$$f(\theta) = -\frac{2m}{\hbar^2 \varkappa} \beta \int_0^\infty e^{-\mu r} \sin(\varkappa r)\, dr$$

$$\frac{e^{i\varkappa r} - e^{-i\varkappa r}}{2i}$$

$$= -\frac{2m}{\hbar^2 \varkappa} \frac{\beta}{2i} \left[\frac{e^{(i\varkappa - \mu)r}}{i\varkappa - \mu} - \frac{e^{(-i\varkappa - \mu)r}}{-i\varkappa - \mu} \right]\Bigg|_0^\infty$$

$$= -\frac{2m}{\hbar^2 \varkappa} \frac{\beta}{2i} \left[\frac{1}{i\varkappa - \mu} + \frac{1}{i\varkappa + \mu} \right]$$

$$\boxed{f(\theta) = -\frac{2m}{\hbar^2} \frac{\beta}{\varkappa^2 + \mu^2}}$$