

Integral form of solution for Schrödinger equation

So far we were avoiding inner scattering region where we have to account for potential V (and we did learn quite a lot)
But let's face the real problem

$$-\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi = E\psi$$

$$\Rightarrow \underbrace{(\nabla^2 \psi + k^2 \psi)}_{\nabla^2 \psi + k^2 \psi = 0} \psi = \frac{2m}{\hbar^2} V\psi = Q, \quad k^2 = \frac{2mE}{\hbar^2}$$

$\nabla^2 \psi + k^2 \psi = 0$ called Helmholtz equation

↑ the one which governs light propagation
because of 0 it is called source free
i.e. light propagation in vacuum

There is a 'trick' for solving above equation with source (Q)

Suppose we can solve

$$(\nabla^2 + \kappa^2) G(\vec{r}) = \delta(\vec{r}) \quad (*)$$

then $\psi(\vec{r}) = \int G(\vec{r} - \vec{r}') Q(\vec{r}') d^3\vec{r}' \quad (**)$

Let's prove it

$$(\nabla^2 + \kappa^2)\psi = \int \underbrace{(\nabla^2 + \kappa^2) G(\vec{r} - \vec{r}')}_{\delta(\vec{r} - \vec{r}') \text{ see } (*)} Q(\vec{r}') d^3\vec{r}' = Q(\vec{r})$$

does not depend on \vec{r}' , so we can move it inside of integral

$G(\vec{r})$ is called the Green's function

and the real trick is to find it.

$$G(\vec{r}) = -\frac{e^{i\kappa r}}{4\pi r}$$

See Griffiths Ch 10.4.1 for this derivation

The rest is easy we plug it to (**)

We get

$$\psi(\vec{r}) = \psi_0 + \int \frac{e^{i k |\vec{r} - \vec{r}'|}}{4\pi |\vec{r} - \vec{r}'|} Q(r') d^3 r'$$

$$\psi(\vec{r}) = \psi_0(\vec{r}) - \frac{2m}{\hbar^2} \int \frac{e^{i k |\vec{r} - \vec{r}'|}}{4\pi |\vec{r} - \vec{r}'|} V(\vec{r}') \psi(\vec{r}') d^3 \vec{r}'$$

, where ψ_0 satisfies source free equation $(\nabla^2 + k^2) \psi_0(\vec{r}) = 0$

There is "slight" problem

$\psi(\vec{r})$ is in LHS and RHS.

So this integral **solution** is **not a solution**

but yet one more equation to solve. (☹)