Homework 03 Name: ____________________

Problem 1 (10 points)
In class we discussed the importance of the dimensionless fine-structure constant

$$\alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}$$

which was used to calculate Hydrogen-like atom relativistic corrections to energy (see chapter 11.5).
Plug the fundamental constants in SI units and calculate α for 3 extra digits.
Do you see a problem? What is missing in the definition? Why is it missing?

Problem 2 (10 points)
Prove Hellmann–Feynman theorem, which states: if Hamiltonian depends on parameter λ and its eigen energy and state depend on λ as as well, i.e. we have $\hat{\mathbf{H}} \left|\psi_\lambda\right\rangle = E_\lambda \left|\psi_\lambda\right\rangle$ then

$$\frac{\mathrm{d}E}{\mathrm{d}\lambda} = \left\langle\psi_\lambda\right| \frac{\mathrm{d}\hat{\mathbf{H}}}{\mathrm{d}\lambda} \left|\psi_\lambda\right\rangle$$

Hint:

$$\frac{\mathrm{d}E}{\mathrm{d}\lambda} = \frac{\mathrm{d}}{\mathrm{d}\lambda} \left\langle\psi_\lambda\right| \hat{\mathbf{H}} \left|\psi_\lambda\right\rangle$$

Problem 3 (10 points)
Using Hellmann–Feynman theorem, calculate

$$\left\langle n, l, m \middle| \frac{1}{r} \middle| n, l, m \right\rangle$$

for Hydrogen-like atom.
Hint:

$$\hat{\mathbf{H}}_r = -\frac{\hbar^2}{2\mu}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2} - Z\frac{e^2}{r}$$

$$\hat{\mathbf{H}}_r \left|R_{nl}\right\rangle = -\frac{\mu e^4 Z^2}{2\hbar^2 (N+l)^2} \left|R_{nl}\right\rangle \tag{1}$$

Problem 4 (10 points)
Calculate the first-order correction to the ground and first excited states of a one dimensional harmonic oscillator due to the relativistic correction to its kinetic energy. The mass of the oscillator is m, and its natural frequency is ω.
What would be an analog of “fine structure constant” in this system?