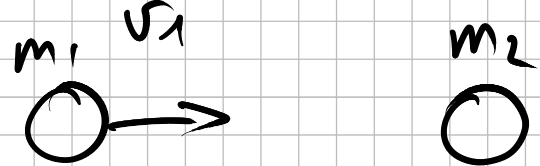
Linear momentum in 3D

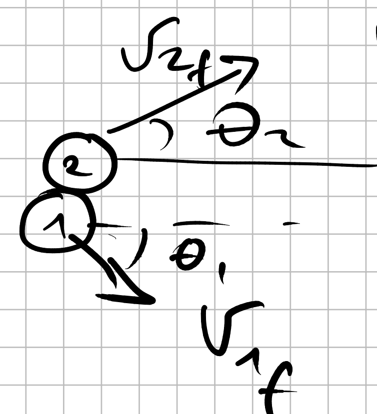
$$\vec{F}_{ext} = \frac{d\vec{P}}{dt} \qquad \vec{P} = \sum_i m_i \vec{v_i}$$

if $\vec{F}_{ext} \Rightarrow \vec{P}$ is a constant
i.e. conserves

Before | After



$\vec{F}_{ext} = 0$



$v_{2fx} = v_{2f} \cdot \cos\theta_2$

$v_{2fy} = v_{2f} \sin\theta_2$

$$x: \quad m_1 \overset{v_1}{\overset{\parallel}{v_{1x_i}}} + m_2 \overset{0}{\overset{\parallel}{v_{2x_i}}} = \quad x: \quad m_1 v_{1x_f} + m_2 v_{2x_f}$$

$$y: \quad m_1 \underset{0}{\underset{\parallel}{v_{1y_i}}} + m_2 \underset{0}{\underset{\parallel}{v_{2y_i}}} = \quad y: \quad m_1 v_{1y_f} + m_2 v_{2y_f}$$

$$x: \quad m_1 v_1 = m_1 v_{1f} \cdot \cos\theta_1 + m_2 v_{2f} \cdot \cos\theta_2$$

$$y: \quad 0 = -m_1 v_{1f} \cdot \sin\theta_1 + m_2 v_{2f} \sin\theta_2$$

Assuming elastic collision

$\Rightarrow$ Energy conserved

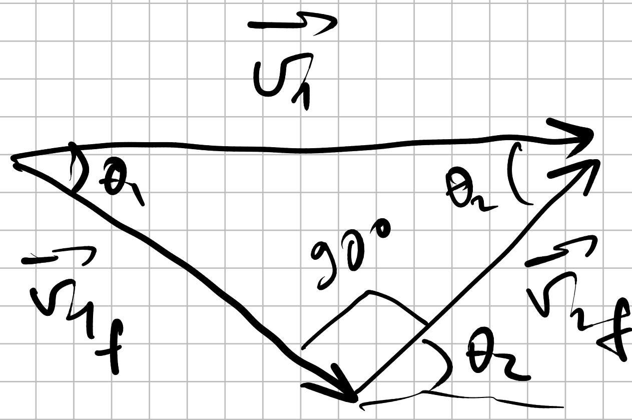
$$\underbrace{m_1 \frac{v_1^2}{2} + \cancel{m_2 \frac{v_2^2}{2}}^{\to 0}}_{\text{before}} = m_1 \frac{v_{1f}^2}{2} + m_2 \frac{v_{2f}^2}{2}$$

Assuming $m_1 = m_2 = m$

$$x: \quad v_1 = v_{1f} \cos\theta_1 + v_{2f} \cdot \cos\theta_2$$

$$y: \quad 0 = -v_{1f} \sin\theta_1 + v_{2f} \sin\theta_2$$

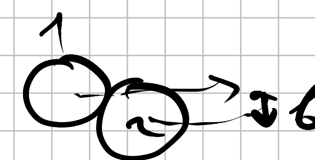
$$\underline{E}: \quad v_1^2 = v_{1f}^2 + v_{2f}^2$$

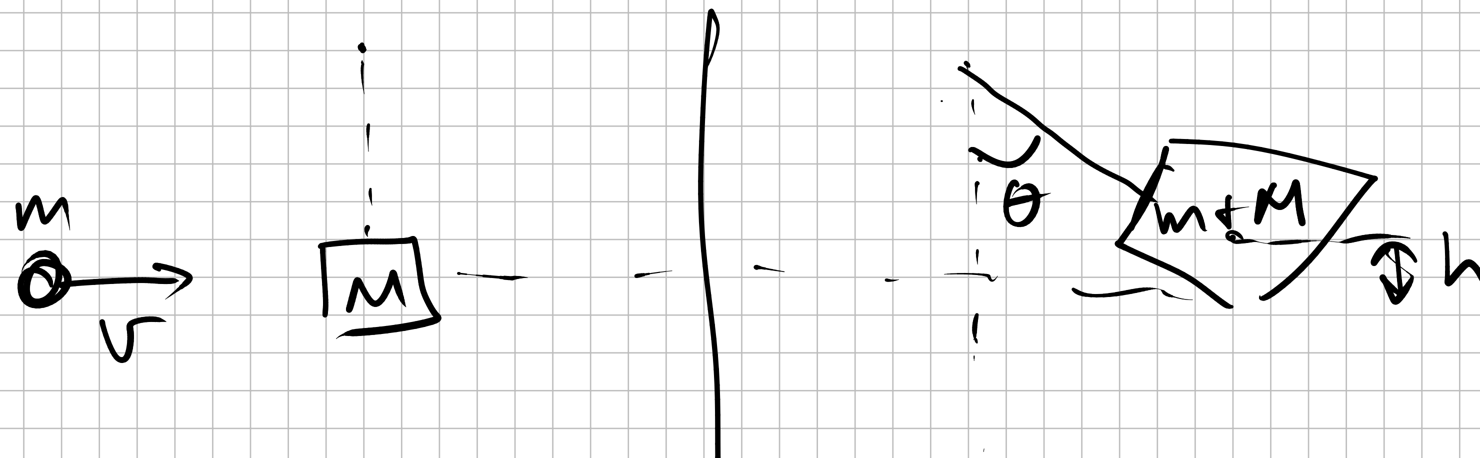


$\Leftarrow$ momentum conservation

$\theta_1 + \theta_2 = 90°$

$$\boxed{\begin{aligned} v_{1f} &= v_1 \cdot \cos\theta_1 = v_1 \sin\theta_2 \\ v_{2f} &= v_1 \cdot \sin\theta_1 = v_1 \cos\theta_2 \end{aligned}}$$





$\Delta t \simeq 0 \Longrightarrow$ $F_{ext} \cdot \Delta t \simeq 0 \Longrightarrow \vec{P}$ conserves

Right after collision

$$x: \quad mv + M\cdot 0 = m v_f + M\cdot v_f$$

embedding the bullet

$$y: \quad 0 + 0 = 0 + 0$$

$$v_f = \frac{mv}{m+M}$$

Collision does not need to preserve Energy

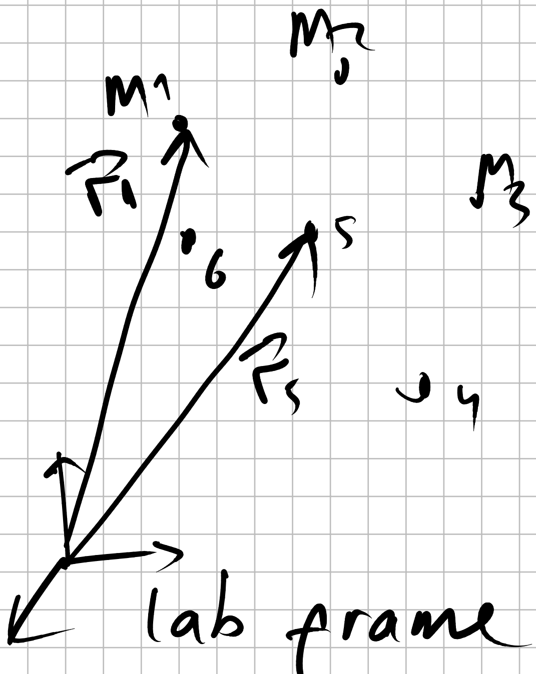
After collision energy preserved

$$\frac{m v_f^2}{2} + \frac{M v_f^2}{2} = (m+M) g h$$

$$\cancel{(m+M)} v_f^2/2 = \cancel{(m+M)} g h$$

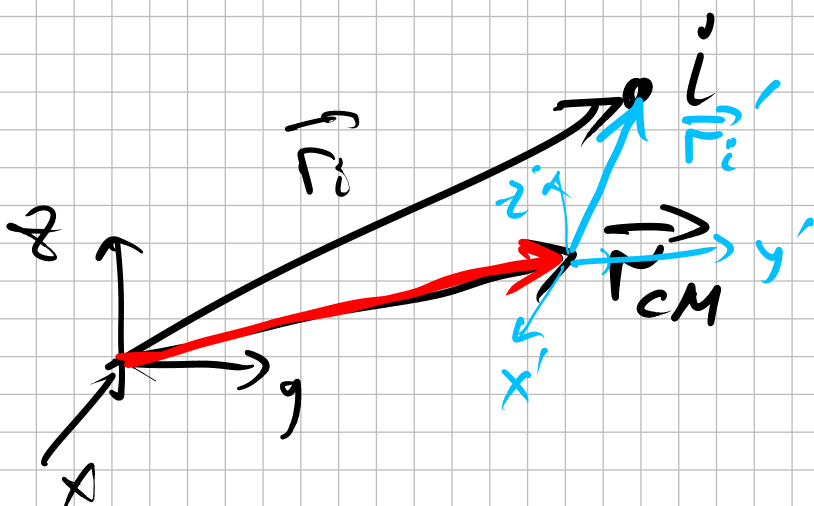
$$\boxed{\left(\frac{m}{M+m} v\right)^2 \frac{1}{2} = g h}$$

Center of mass



$$\boxed{\begin{aligned} M_{tot} &= \sum_i m_i \\ \vec{r}_{CM} &= \frac{\sum_i m_i \vec{r}_i}{M_{tot}} \end{aligned}}$$

center of mass position



$$\vec{r}_i = \vec{r}_{CM} + \vec{r}\,'$$

$$\vec{F}_{\substack{net\\ext}} = \frac{d\vec{P}}{dt} = \frac{d}{dt}\left(\sum_i m_i \vec{v}_i\right) =$$

$$= \frac{d}{dt}\left(\sum_i m_i \frac{d\vec{r}_i}{dt}\right) =$$

$$= \frac{d}{dt}\cdot\frac{d}{dt}\left(\sum_i m_i \vec{r}_i\right)\frac{M_{tot}}{M_{tot}} =$$

$\vec{r}_{cM}$

$$= M_{tot}\cdot\frac{d^2}{dt^2}\vec{r}_{cM} =$$

$$\boxed{\vec{F}_{\substack{net\\ext}} = M_{tot}\,\vec{a}_{cM}}$$