AC signals and filters.

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WILLIAM & MARY
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Power dissipation

Recall that power dissipated by element is

$$P = VI$$

where V and I are real.
Since we use a substitute
$V\cos(\omega t) \to Ve^{i\omega t}$ and $I\cos(\omega t) \to Ie^{i\omega t}$,
we need to write

$$P = Re(V)Re(I)$$

Recall the Ohm's law

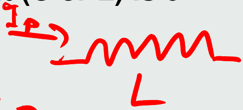
$$V = ZI$$

Power dissipation by a reactive element

Theorem

Average power dissipated by a reactive element (C or L) is 0

Lets use as example an inductor.



$$Z_L = i\omega L = e^{i\frac{\pi}{2}}\omega L,\ I_L = I_p e^{i\omega t}$$

$$V_L = Z_L I_L = e^{i\frac{\pi}{2}}\omega L I_L = \omega L I_p e^{i(\omega t + \frac{\pi}{2})}$$

$$Re(I_L) = I_p \cos(\omega t),\ Re(V_L) = -\omega I_p L \sin(\omega t)$$

Im

$i \cdot e^{i\omega t} = i\cos(\omega t) + (i \cdot i)\sin(\omega t)$

-1

Thus average power dissipated by the inductor

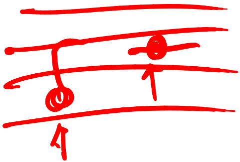
$$P = \int_0^T Re(I_L) Re(V_L) dt = -\int_0^T I_p \cos(\omega t)\omega I_p L \sin(\omega t) dt$$

$$P = -\omega I_p^2 L \int_0^T \cos(\omega t)\sin(\omega t) dt = \omega I_p^2 L \int_0^T \frac{1}{2}\sin(2\omega t) dt = 0$$

Fourier transform

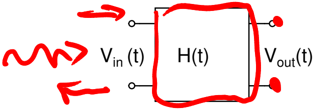
If function $f(t)$ goes to zero at $\pm\infty$ then $\hat{f}(\omega)$ exists such as

$$f(t) = \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} d\omega$$



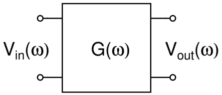
Transfer function

Time domain



$$V_{out}(t) = \int_{-\infty}^{t} H(t-\tau)V_{in}(\tau)d\tau$$

Frequency domain $V_{in} = V_{in}(\omega) \cdot e^{i\omega t}$



$$V_{out}(\omega) = G(\omega)V_{in}(\omega)$$

Where G is complex transfer function or gain.

Definition

$$G(\omega) = \frac{V_{out}(\omega)}{V_{in}(\omega)} = |G(\omega)|e^{i\phi(\omega)}$$

Often used values of G in dB

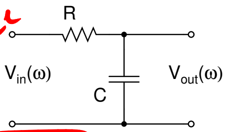
$$dB = 20\log_{10}(|G(\omega)|)$$

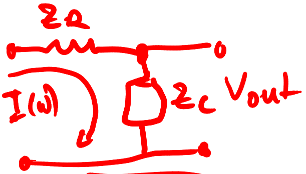
Simple example: RC low-pass filter

$Z_c = \frac{1}{i\omega C}$

$I = \frac{V_{in}(\omega)}{Z_R + Z_c}$

$V_{out} = I(\omega) Z_c$





$$G(\omega) = \frac{V_{out}(\omega)}{V_{in}(\omega)} = \frac{\frac{1}{i\omega C}}{R + \frac{1}{i\omega C}} = \frac{1}{i\omega RC}\frac{1}{1 + \frac{1}{i\omega RC}} = \boxed{\frac{1}{1 + i\omega RC}}$$

$1/RC = \omega_{3dB}$

defining $\omega_{3dB} = \frac{1}{RC}$

$$G(\omega) = \boxed{\frac{1}{1 + i\frac{\omega}{\omega_{3dB}}}} = \frac{1}{\sqrt{1 + \frac{\omega^2}{\omega_{3dB}^2}}} e^{i\phi} \quad \phi = \arctan(-\frac{\omega}{\omega_{3dB}})$$

$\approx |G|$

$\omega = 1/RC = \omega_{3dB}$

Note

$$|G(\omega = \omega_{3dB})| = 20\log_{10}\left(\frac{1}{\sqrt{1+1}}\right) = 20\log_{10}\left(\frac{1}{\sqrt{2}}\right) = -3dB$$

$$Z_C = \frac{1}{i\omega C}$$

$\omega \to 0$,

$Z_C \to \infty$

$$V_{out} = V_{in} \frac{Z_C}{\cancel{Z_R}+Z_C} = V_{in} \Rightarrow G = 1$$

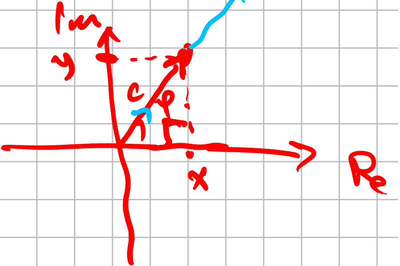
0 $\quad$ $\to 1$

$\omega \to \infty$, $Z_C = \frac{1}{i\omega C} \to 0$

$V_{out} = 0$, $G(\omega \to \infty) = 0$

$$G = \frac{1}{1+i\frac{\omega}{\omega_{3dB}}} \cdot \frac{1-i\,\omega/\omega_{3dB}}{1-i\,\omega/\omega_{3dB}} =$$

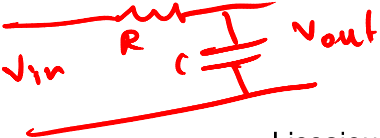
$$= \frac{1-i\,\omega/\omega_{3dB}}{1-\underbrace{(i)^2}_{-1}\frac{\omega^2}{\omega_{3dB}^2}} = \frac{\overset{x}{1} - \overset{y}{i\frac{\omega}{\omega_{3dB}}}}{1+\frac{\omega^2}{\omega_{3dB}^2}}$$



$$c = x + iy = A \cdot e^{i\varphi}$$

$$A = \sqrt{x^2+y^2} \qquad \tan\varphi = \frac{y}{x}$$

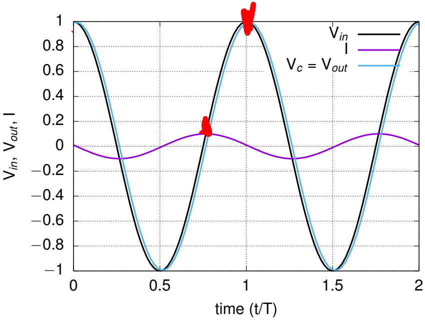
RC low-pass filter at $\omega = .1/RC$ $< 0.1\,\omega_{3dB}$



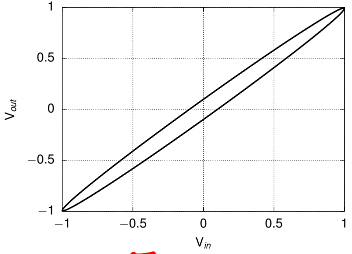
$$V_c = Z_c \cdot I_c$$

$$\frac{i}{i}\,\frac{1}{i\omega C} = \frac{-i}{\omega C}$$

Signal vs time



Lissajous figure

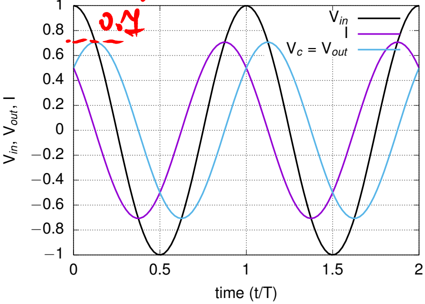


$$-i = e^{-i\frac{\pi}{2}}$$

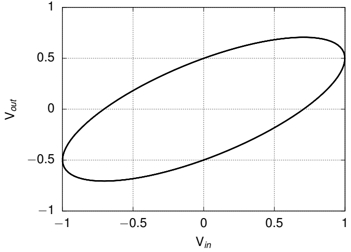
RC low-pass filter at $\omega = 1/RC$ $= \omega_{3dB}$

Signal vs time $1/\sqrt{2}$

≈ 0.7

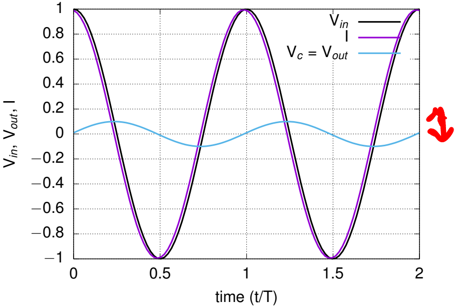


Lissajous figure

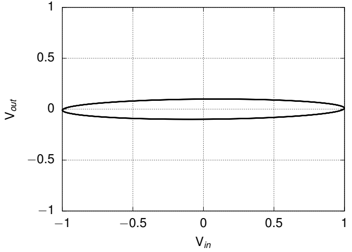


RC low-pass filter at $\omega = 10/RC$ = 10 ω_{3dB}

Signal vs time



Lissajous figure

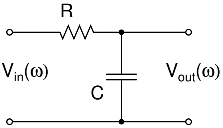


Bode plots

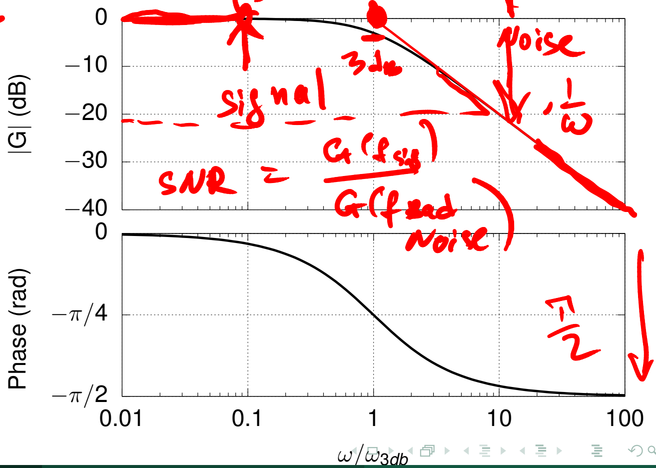
Definition

Bode plot: plots of magnitude and phase of the transfer function, where $|G|$ is often plotted in dB

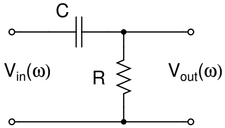
$\partial = 20 \cdot \log_{10} 1$

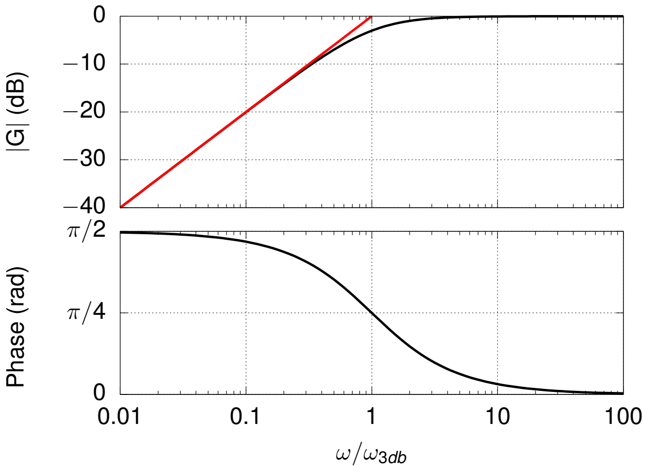


$$G(\omega) = \frac{1}{1 + i\frac{\omega}{\omega_{3dB}}}$$



RC high-pass filter



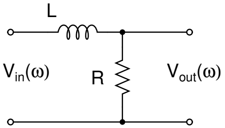


$$G(\omega) = \frac{V_{out}(\omega)}{V_{in}(\omega)} = \frac{R}{R + \frac{1}{i\omega C}} = \frac{i\omega RC}{1 + i\omega RC} = \frac{i\frac{\omega}{\omega_{3dB}}}{1 + i\frac{\omega}{\omega_{3dB}}}$$

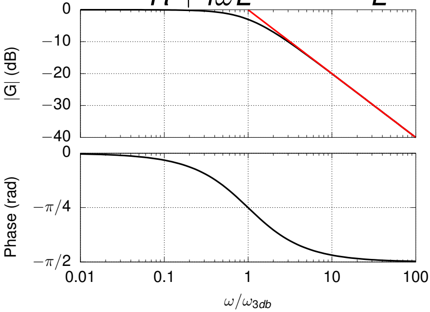
with $\omega_{3dB} = \frac{1}{RC}$

RL filters

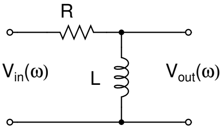
RL low-pass filter



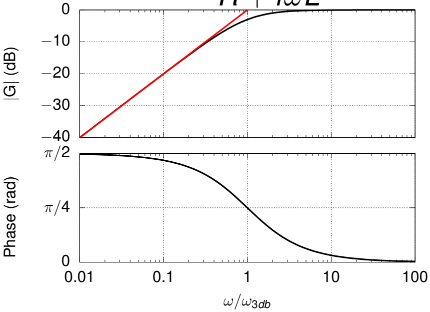
$$G(\omega) = \frac{R}{R + i\omega L}, \omega_{3dB} = \frac{R}{L}$$



RL high-pass filter



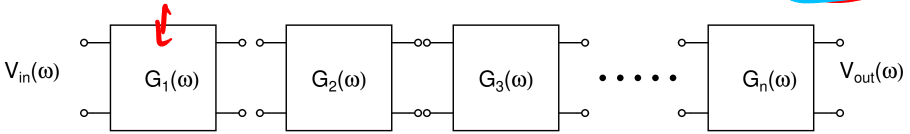
$$G(\omega) = \frac{i\omega L}{R + i\omega L}$$



Filters chain

$Z_2 \cong 10 Z_1$

$Z_1 \ll Z_2$



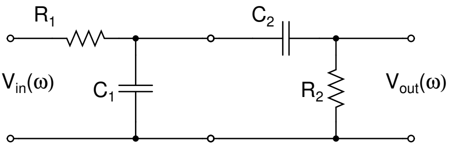
Technically next stage loads the previous and it is quite hard to calculate total transfer function.
However if we use rule of 10 to avoid overloading the previous filter.
Every next stage resistor $R_{i+1} > 10R_i$ we can approximate

$$G_t(\omega) \approx G_1(\omega)G_2(\omega)G_3(\omega)\cdots G_n(\omega)$$

$R \parallel R_L$ 10R

$= \frac{R \cdot R_L}{R + R_L} \approx R$

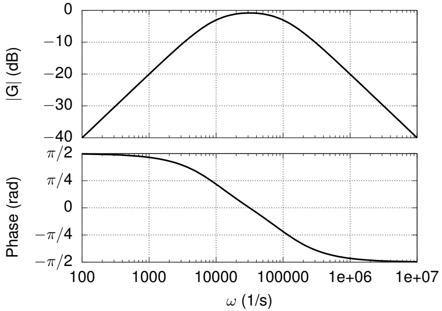
~10R



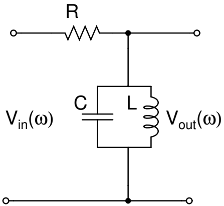
$$G_t(\omega) \approx G_1(\omega) G_2(\omega)$$

$$G_t(\omega) \approx \frac{1}{1 + i\frac{\omega}{\omega_{1_{3dB}}}} \frac{i\frac{\omega}{\omega_{2_{3dB}}}}{1 + i\frac{\omega}{\omega_{2_{3dB}}}}$$

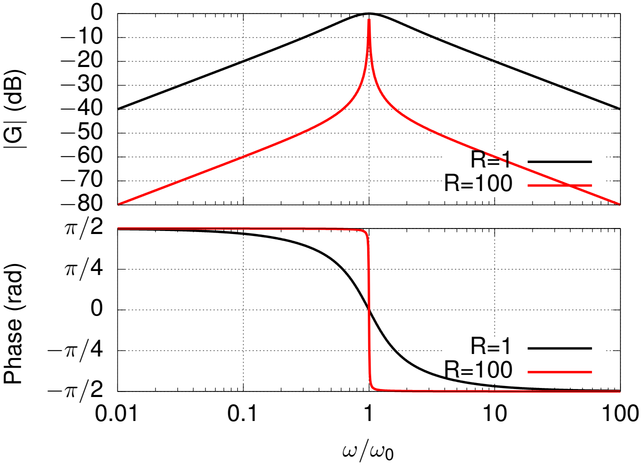
For $R_1 = 1k\Omega, R_2 = 100k\Omega,$
$C_1 = C_2 = .01\mu F$



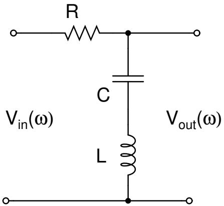
RLC band pass filter



$$\omega_0 = \frac{1}{\sqrt{LC}}$$



Notch filter - Band stop filter



$$\omega_0 = \frac{1}{\sqrt{LC}}$$