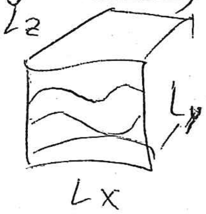
lecture 30 — White dwarfs and neutron stars

Compact object with masses $\approx M_\odot$ and $R < R_\odot/100$, very hot though

$T \gtrsim 40'000\,K$

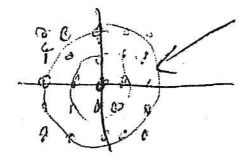
Degeneracy pressure



$$N_x \frac{\lambda_x}{2} = L_x$$

$$\lambda_x = \frac{2 L_x}{N_x}$$

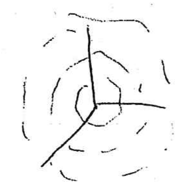
$L_x = L_y = L_z$



constant E surface

$$p_{x,y,z} = \frac{h}{\lambda} = \frac{h N_{x,y,z}}{2 L_{x,y,z}}$$

$$\varepsilon = \frac{p_x^2 + p_y^2 + p_z^2}{2m} = \frac{h^2}{8m}\frac{1}{L^2}\left(N_x^2 + N_y^2 + N_z^2\right)$$



number of possible sits with $N^2 = N_x^2 + N_y^2 + N_z^2$

$$\# = \frac{4\pi}{3}\frac{N^3}{1^3}$$

volume of cell

since $N_x, N_y, N_z > 1 \implies$ we need to take $\frac{1}{8}$ of it i.e. 1 octant

So $N_f = 2 \cdot \left(\frac{1}{8}\right)\left(\frac{4\pi}{3}\frac{N_{max}^3}{1}\right)$

N_f: # fermions; 2: spin $\pm 1/2$

So maximum energy occupied by fermion

$$N_{max} = \left(\frac{3 N_f}{\pi}\right)^{1/3} \implies \varepsilon_{max} = \varepsilon_F = \frac{h^2}{8m}\frac{1}{L^2}\cdot\left(\frac{3 N_f}{\pi}\right)^{2/3}$$

N_{max}: quant. number; ε_F: Fermi

$$= \frac{h^2}{8m}\left(\frac{3}{\pi}\frac{N_f}{L^3}\right)^{2/3}$$

$\frac{N_f}{L^3}$: n_f density

$$\varepsilon_F = \frac{1}{2}\overbrace{\left(\frac{h^2}{2^2 m \pi^2}\right)}^{\hbar^2}\pi^2\left(\frac{3}{\pi} n_f\right)^{2/3}$$

$$= \frac{\hbar^2}{2m}\left(3\pi^2 n_f\right)^{2/3}$$

maximum energy in degenerate gas of Fermions

Let's consider electron gas

$$n_f = n_e = \underset{\substack{\uparrow \\ \#p + \#n}}{\frac{\overset{\text{charge}}{Z}}{A}} \cdot \frac{\rho}{\underset{\text{nucleon mass}}{m_H}}$$

$m_H \simeq m_n \simeq m_p$

$$\varepsilon_{F_e} = \frac{\hbar^2}{2m}\left(3\pi^2\left(\frac{Z}{A}\right)\frac{\rho}{m_H}\right)^{2/3}$$

Let's consider typical energy per particle and in fermion gas

Strictly speaking we need to find total energy of the gas

$$\int_0^{\varepsilon_F} n(E)\, E\, dE$$

but let's say that each fermion has $\simeq \varepsilon_F$.

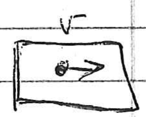
this is for alternative way to derive $P_F \cdot V = E_{total} \cdot \frac{2}{3}$

which is not used in class

Degeneracy pressure due to fermions

non relativistic
$v = p/m_f$

$$V \cdot P_f = \sum_p \frac{1}{3} p \cdot v \cdot = \sum_p \frac{1}{3} p \cdot \frac{p}{m_f} \quad \text{fermion mass}$$



$$= \sum_p \frac{1}{3}\frac{p^2}{\cancel{2}m_f} = \frac{2}{3}\frac{h^2}{8m}\frac{1}{L^2}\sum_{n_x,n_y,n_z}(n_x^2+n_y^2+n_z^2) =$$

$$= \frac{1}{3}\frac{h^2}{8m_f}\frac{2}{L^2}\int_0^{N_{max}} N^2 \cdot \underbrace{4\pi N^2 dN}_{\text{shell volume}} =$$

$$= \frac{2}{3}\frac{h^2}{8m_f}\frac{4\pi}{5}N^5 = \frac{2}{3}\frac{h^2}{8m_f}\frac{1}{L^2}\frac{4\pi}{5}\left(\frac{3N_f}{\pi}\right)^{5/3}$$

$$P_f V = P_f L^3 = \frac{\cancel{2}}{3}\frac{h^2}{m_f}\frac{\cancel{8}\pi}{5}\frac{1}{L^2}\left(\frac{3N_f}{\pi}\right)^{5/3}$$

$$P_f = \frac{\cancel{4}}{3}\frac{h^2}{m_f}\frac{\pi}{5}\left(\frac{3}{\pi}\left(\frac{N_f}{L^3}\right)\right)^{5/3} =$$

$$= \underbrace{\left(\frac{h}{2\pi}\right)^2}_{\hbar^2}\frac{1}{m_f}\frac{\pi\cdot 4\pi^2}{3\cdot 5}\left(\frac{3}{\pi}n_f\right)^{5/3} =$$

$$= \frac{\hbar^2}{m_f}\frac{1}{5}\pi^{4/3}\cdot 3^{2/3}\cdot n_f^{5/3}$$

$$\boxed{P_f = \underbrace{\frac{\hbar^2}{m_f}\frac{(\pi^2\cdot 3)^{2/3}}{5}}_{K} n_f^{5/3}}$$

non relativistic

$$n_f = \frac{Z}{A}\frac{\rho}{m_H}$$

Condition of degeneracy, when degenerate pressure kicks in

thermal energy per particle

$$\frac{3}{2} kT < \mathcal{E}_F$$

$$\frac{3}{2} kT < \frac{\hbar^2}{2m_e}\left(3\pi^2\left(\frac{Z}{A}\right)\frac{\rho}{m_H}\right)^{2/3}$$

$$\boxed{\frac{T}{\rho^{2/3}} < \frac{\hbar^2}{3m_e k}\left(3\pi^2\left(\frac{Z}{A}\right)\frac{1}{m_H}\right)^{2/3}}$$

White dwarf must have core of heavy elements $\Rightarrow N_p \approx N_n$ (proton, neutron)

$$\Rightarrow \frac{Z}{A} = \frac{N_p}{N_p + N_n} = \frac{1}{2}$$

$$\boxed{\frac{T}{\rho^{2/3}} < D = 1261 \ K \frac{m^2}{kg^{2/3}}}$$ for electrons

Sun ~~White dwarf~~ $T_c \simeq 1.5 \cdot 10^7$

$\rho_c \simeq 1.5 \cdot 10^5 \ kg/m^3$

$$\frac{T_c}{\rho_c^{2/3}} = 5500 \ K \frac{m^2}{kg^{2/3}}$$

so degeneracy for the sun does not kick in

But for White dwarf

$$\frac{T_c}{\rho_c^{2/3}} < 100$$ so they are degenerate

$T_c \simeq 7.6 \cdot 10^7$, $\rho_{aver} \simeq 3 \cdot 10^9 \frac{kg}{m^3}$

The Chandrasekhar limit

Gravitational pull ⇒ hydrostatic pressure = gas pressure

$$\frac{2}{3}\pi G \rho^2 R^2 = K\left(\frac{Z}{A}\frac{\rho}{m_H}\right)^{5/3}$$

$$R^2 = \frac{K}{\frac{2}{3}\pi G \rho^2}\left(\frac{Z}{A}\frac{\rho}{m_H}\right)^{2/3}$$

$$R^2 = \dots \frac{1}{\rho^2}\left(\frac{Z}{a}\frac{\rho^?}{m_H}\right)^{2/3} =$$

$$= \dots \left(\frac{Z}{a}\frac{1}{m_H}\right)\rho^{-2/3} \qquad \left(\frac{M}{\frac{4\pi}{3}R^3}\right)^{-2/3}$$

$$\rho^2 R^2 \sim \rho^{5/3}$$

$$\left(\frac{M}{R^3}\right)^2 \sim \rho^{5/3} = \left(\frac{M}{R^3}\right)^{5/3}$$

$$\frac{M^{2-5/3}}{R^{6-5}} = \frac{M^{1/3}}{R} \Rightarrow = \text{const}$$

$$M \cdot R^3 \Rightarrow \boxed{M \cdot V = \text{const}}$$

General for any fermionic gas

$$M\uparrow \Rightarrow V\downarrow \Rightarrow n_f\uparrow$$

$\frac{mv^2}{2} = \varepsilon_f \sim n_f^{2/3}$ so v will be $> c$ which is impossible

Relativistic gas

$\varepsilon_f = \frac{\hbar^2}{2m}(3\pi^2 n_e)^{2/3} = \frac{mv^2}{2}$

$v \simeq c$

$n_e \nearrow \; v_e \nearrow \; \leq c$

$E = c \cdot p$ alternative way

More over $PV = \frac{E}{3}$ (same as for photons!)

$$\boxed{P = \frac{(3\pi^2)^{1/3}}{4} \hbar c \, (n_e)^{4/3}}$$

see non relativistic case and use

$V \cdot P = \sum_p \frac{1}{3} p \cdot c$

replaced v

Hydrostatic

derivation to be done in HW

$\rho^2 R^2 \sim \rho^{4/3}$

$\frac{M^2}{R^6} R^2 \frac{R^4}{M^{4/3}} = \text{const}$

So there is a limit for mass

$\boxed{M^{2/3} = \text{const}} =$

$$M_{ch} = \left(\frac{\frac{(3\pi^2)^{1/3}}{4} \hbar c \left(\frac{Z}{A}\frac{1}{m_H}\right)^{4/3}}{\frac{2}{3}\pi G} \right)^{3/2}$$

limit for a white dwarf

$$M_{ch} = \left(\frac{\hbar c}{G}\right)^{3/2} \left(\frac{Z}{A}\frac{1}{m_H}\right)^2 \frac{3\sqrt{2\pi}}{8}$$

$\boxed{M_{ch} \approx 1.4 \, M_\odot}$ with extra corrections