 lecture 9

Black body radiation

Black body – absorbs all energy, thermolizes, emits thermal radiation

<example spectrum>

y-axis how much enery is emitted per unit wavelength

Wien's law (1893) explained (derived the thermodyn. argument) empirical law)

spectral energy density $= \frac{\text{energy}}{\text{volume} \cdot \text{wavelength}} = \frac{f(\lambda T)}{\lambda^5}$

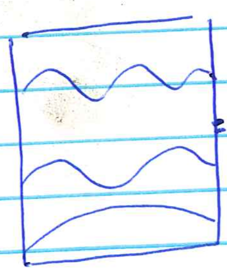
$b \approx 2.9 \cdot 10^{-3}$ m·K

Nobel prize, 1911

where $f(\lambda T)$ – universal function

$\lambda_{max} T = b$ – Wien's displacement constant

Model of a black body – closed conducting cavity



Because of the boundary conditions → standing waves

1D: $n \cdot \frac{\lambda_n}{2} = L \Rightarrow \lambda_n = \frac{2L}{n}$

~~wave vector~~ $\nu_n = \frac{c}{\lambda} = \frac{cn}{2L}$

How many modes?

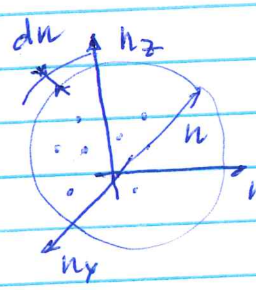
$\Delta k = \frac{\pi}{L} \Delta n$

3D: Standing waves in x, y, z

$k_x = n_x \pi / L_x, \quad k_y = n_y \pi / L_y, \quad k_z = n_z \pi / L_z$

total wavevector

$2\pi\nu / c = k = \sqrt{k_x^2 + k_y^2 + k_z^2} = \frac{\pi}{L} \underbrace{\sqrt{n_x^2 + n_y^2 + n_z^2}}_{n}$



Number of modes for wavenumber values $k = |\vec{k}| \div k + dk$

⇓

area of the spherical layer

$N(n)dn = 4\pi n^2 dn \cdot \frac{1}{8} \cdot 2$ ↑polarization

(to account for $k_{x,y,z}$ all > 0) → $\frac{1}{8}$

~~spectral~~ mode density

$$\frac{\text{Number of modes}}{\text{volume} \cdot d\nu} = \frac{\frac{\pi}{\cancel{8}}\left(\frac{2L}{c}\right)^3 \nu^2 d\nu}{V} = \frac{8\pi}{c^3}\nu^2 d\nu$$

Energy density

$$u_\nu d\nu = \langle \substack{\text{energy}\\ \text{per mode}} \rangle \cdot \text{mode density}$$

Classical physics → Boltzman distribution

$$\langle E \rangle = \frac{\int_0^\infty E e^{-E/k_BT} dE}{\int_0^\infty e^{-E/k_BT} dE} = kT \frac{\int_0^\infty x e^{-x} dx}{\int_0^\infty e^{-x} dx} = k_BT$$

equipartition of energy

$$u_\nu d\nu = \frac{8\pi\nu^2}{c^3} k_BT \, d\nu$$

in terms of $\lambda = \frac{c}{\nu}$: $d\nu = -\frac{c}{\lambda^2} d\lambda$

$$u_\lambda d\lambda = -u_\nu d\nu = \frac{8\pi}{c^3}\frac{c^2}{\lambda^2} k_BT \left(\frac{c}{\lambda^2}\right) d\lambda$$

$$u_\lambda d\lambda = 8\pi k_B \cdot \frac{\lambda T}{\lambda^5} d\lambda$$

Rayleigh-Jeans expression

follows Wien's law

Works well for large λ, diverges at small λ

$$\int_0^\infty u_\lambda d\lambda = 8\pi k_BT \int_0^\infty \frac{d\lambda}{\lambda^4} = \frac{8\pi k_BT}{3} \frac{1}{\lambda^3}\Big|_0^\infty \rightarrow$$

total amount of radiating energy diverges

Plank's quantization

Energy of each mode is quantized, namely it can only have values

$$\varepsilon_n = n h\nu$$

In this case the average energy per mode

$$\langle E \rangle = \frac{\sum_{n=0}^{\infty} \varepsilon_n e^{-\varepsilon_n/k_BT}}{\sum_{n=0}^{\infty} e^{-\varepsilon_n/k_BT}} = \cancel{k_BT}\, \frac{h\nu\, \cancel{\frac{h\nu}{k_BT}} \sum_{n=0}^{\infty} n\, e^{-\hbar\nu \cdot n/k_BT}}{\sum_{n=0}^{\infty} e^{-(\hbar\nu/k_BT)\cdot n}}$$

$$\langle E \rangle = \cancel{k_BT\alpha}\, h\nu \frac{\sum_{n=0}^{\infty} n\, e^{-\alpha n}}{\sum_{n=0}^{\infty} e^{-\alpha n}} \qquad \alpha = \frac{h\nu}{k_BT}$$

$$\sum_{n=0}^{\infty} e^{-\alpha n} = \frac{1}{1-e^{-\alpha}}$$

$$\sum_{n=0}^{\infty} n\, e^{-\alpha n} = -\frac{\partial}{\partial\alpha} \sum_{n=0}^{\infty} e^{-\alpha n} = -\frac{\partial}{\partial\alpha}\left(\frac{1}{1-e^{-\alpha}}\right) =$$

$$= \frac{e^{-\alpha}}{(1-e^{-\alpha})^2}$$

$$\langle E_\nu \rangle = h\nu \frac{e^{-\alpha}}{1-e^{-\alpha}} = \hbar\nu \frac{1}{e^{\frac{h\nu}{k_BT}} - 1}$$

$$\langle E_\nu \rangle \to k_BT \quad \text{for } \nu \to 0$$

$$\to \hbar\nu\, e^{-\hbar\nu/k_BT} \qquad \nu \to \infty$$

$$u_\nu \, d\nu = \frac{8\pi}{c^3} \nu^2 \cdot h\nu \frac{1}{e^{h\nu/k_B T} - 1} \, d\nu =$$

$$= \frac{8\pi h}{c^3} \nu^3 \frac{1}{e^{h\nu/k_B T} - 1} \, d\nu$$

and rewriting it through λ s

$$u_\lambda \, d\lambda = \frac{8\pi hc}{\lambda^5} \frac{1}{e^{hc/\lambda k_B T} - 1} \, d\lambda$$

energy density states "inside the box"

The radiation flux (intensity)

$$\frac{1}{2}\langle \cos^2\theta \rangle = \frac{1}{4}$$

$$I = \frac{dE}{dt \cdot dA} = \frac{dE}{\frac{dx}{c} \cdot dA} = c \cdot \frac{dE}{dV} = c \cdot u$$

$$I_\lambda \, d\lambda = \frac{c \cdot u_\lambda}{4} \, d\lambda = \frac{8\pi h c^2}{4 \cdot \lambda^5} \frac{1}{e^{hc/\lambda k_B T} - 1} \, d\lambda$$

Total emitted radiation

$$I(T) = \int_0^\infty I_\lambda \, d\lambda =$$

$$= \int_0^\infty \frac{8\pi hc^2}{4} \frac{1}{e^{hc/\lambda k_B T} - 1} \frac{d\lambda}{\lambda^5} = \left\{ \begin{array}{l} x = \frac{hc}{\lambda k_B T} \\ \lambda = \frac{hc}{x k_B T} \\ d\lambda = -\frac{hc}{x^2 k_B T} dx \end{array} \right.$$

$$= \frac{8\pi hc^2}{4} \int_0^\infty \frac{1}{e^x - 1} \frac{hc}{x^2 k_B T} dx \cdot x^5 \left(\frac{k_B T}{hc}\right)^5 =$$

$$= \frac{8\pi hc^2}{4} \left(\frac{k_B T}{hc}\right)^4 \underbrace{\int_0^\infty \frac{x^3 \, dx}{e^x - 1}}_{\pi^4/15}$$

note simplier to do with $\int_0^\infty I_\nu \, d\nu = I(T)$

Total emitted intensity

$$I(T) = \underbrace{\left(\frac{2\pi^5 k_B^4}{15 h^3 c^2}\right)}_{\sigma \,=\, 5{,}67\cdot 10^{-8} \frac{W}{m^2K^4}} T^4$$

Stephan - Boltzman constant

Luminocity of the star

$$L = 4\pi R^2 \cdot I = 4\pi R^2 \cdot \sigma T^4$$