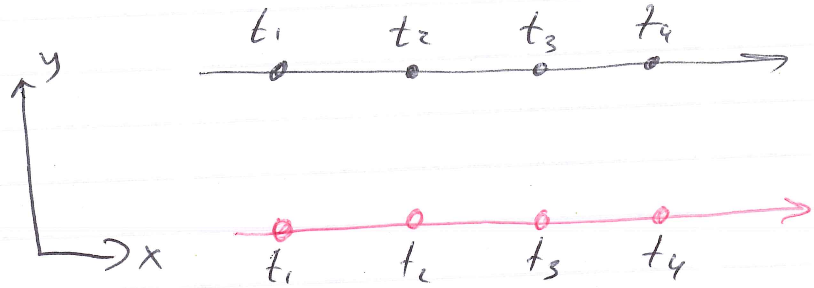
Lecture 07

Some extra trivia:

Q1: For this trajectories, are bodies gravitationally bounded

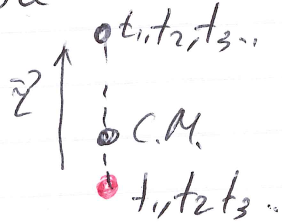


A_1: naive no! Because C.M. seems to move

A_2: But we are not in C.M. ref. frame!

C.M. must be on a line connecting 2 bodies

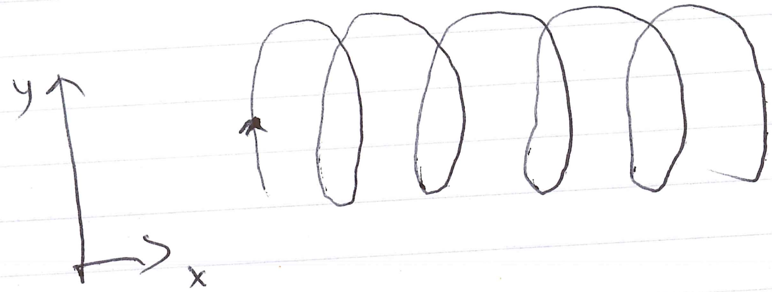
Possible C.M. ref frame, recall $\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$



So there is no apparent orbit around C.M.

$\boxed{\text{So either } |r| = \infty \text{ or they are not gravitationally bounded}}$

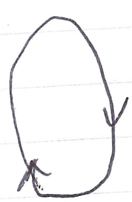
Q2:
what if we can track one body which does this.



Q2a: Is it even possible?

A: Yes, if C.M. moves along X with constant velocity Which is totally fine

So in C.M. ref. frame it will look like



Q2B: can we say where is C.M?

A2B: No! There are 2 possible locations:

Without observing 2nd body we have no clue.

Or we need to time 1st body orbit. It moves faster near "perihelion" i.e. focus.

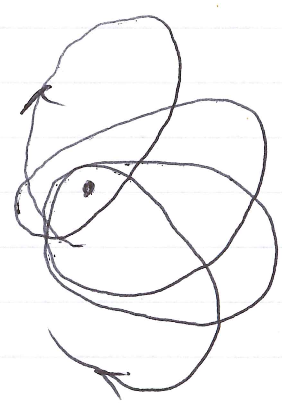
Q3: What if more than 2 ~~gravitationally bounded~~ bodies,

A: The problem has no analytical solution ☹

Bring your computer and solve it numerically (some might remember my midtern in Phys 256)

If 3rd mass relatively light, we can treat it as perturbation

so original orbit (2 body case) will **precess**



This is mostly the case for Sun-planets orbits in our solar system.

Halley's comet problem (2.14 from the book)

$T = 76$ years ← orbital period
$e = 0.9673$

Find 'a'

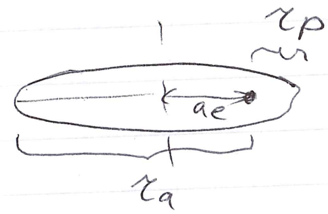
Well 3rd Kepler's law

$T^2 = a^3$ Note so far experimental observ. only

in years; in au
note that by Kepler's law this is average distance

$\Rightarrow$

$a \approx \sqrt[3]{76^2} = 18$ au



$b \approx a \cdot \sqrt{1-e^2} = 4.55$ au

aphelion distance from sun
$r_a = a + ae = a(1+e) \approx 35.3$ AU

perihelion $r_p = a - ae = a(1-e) \approx 0.59$ AU

Ok, what about mass of the system?

Naively: let's use 'e' expession...

But 'e' is a bad parameter: many orbits have the same 'e'.

Think about circular orbits ($e=0$) around heavy body.

So to find masses, we need to go ~~deeper~~ deeper into 3rd Kepler's law derivation,

Recap

$$E = \frac{\mu}{2}(\dot{r})^2 + \frac{L^2}{2\mu r^2} - \frac{\alpha}{r}$$

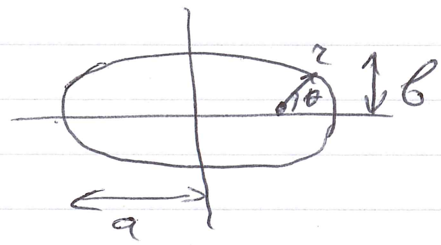
$$L = \mu r^2 \dot{\theta} = 2\mu(\dot{A})$$

$$\frac{p}{r} = 1 + e\cos\theta$$

$$a = \frac{p}{1-e^2} \; ; \; b = \frac{p}{\sqrt{1-e^2}}$$

$$b^2 = a^2(1-e^2)$$

~~ell~~ $$p = \frac{L^2}{\mu\alpha^2}$$



3rd Kepler's law

(Or how ~~we~~ do we know the mass of the sun?)

Recall that

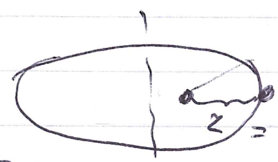
$$L = \mu r^2 \dot{\theta} = \mu 2 (d\dot{A})$$

$$\Rightarrow \int dA = \int \frac{L}{2\mu} \cdot dt$$

period

$$\pi a \cdot b = \frac{L}{2\mu} \cdot T$$

perihelion



$$\pi \frac{p}{1-e^2} \cdot \frac{p}{\sqrt{1-e^2}} = \frac{L}{2\mu} T$$

$$\pi a b = \pi a^2 \sqrt{1-e^2} = \cancel{\pi a^2 \frac{2EL^2}{\mu\alpha^2}} = \frac{L}{2\mu} T$$

looking for L and e from energy conservation p.v.

$$E = \frac{\mu (\dot{r})^2}{2} + \frac{L^2}{2\mu r^2} - \frac{\alpha}{r}$$

$(\dot{r}) = 0$ in perihelion

$$\Rightarrow E = \frac{L^2}{2\mu r^2} - \frac{\alpha}{r} \qquad (r = a(1-e))$$

$$\Rightarrow E = \begin{cases} \dfrac{L^2}{2\mu a^2 (1-e)^2} - \dfrac{\alpha}{a(1-e)} \\ \dfrac{L^2}{2\mu} \dfrac{1}{a^2(1+e)^2} - \dfrac{\alpha}{a(1+e)} \end{cases}$$ aphelion

$$T = \cancel{2\pi a^2 \frac{2EL}{\alpha^2}}$$

$$\frac{L^2}{2\mu a^2}\frac{1}{(1-e)^2}-\frac{\alpha}{\cancel{a}(1-e)}=\frac{L^2}{2\mu a^2}\frac{1}{(1+e)^2}-\frac{\alpha}{\cancel{a}(1+e)}$$

$$\frac{L^2}{2\mu a}\frac{1}{1-e}\left(\frac{1}{1-e}-\alpha\right)=\frac{L^2}{2\mu a}\frac{1}{1+e}\left(1/_{1+e}-\alpha\right)$$

$$\cancel{\frac{L^2}{2\mu a}\frac{e}{(1-e)^2}=\frac{L^2}{2\mu a}\frac{(-e)}{(1+e)^2}}$$

$$\frac{L^2}{2\mu a}\left[\frac{1}{(1-e)^2}-\frac{1}{(1+e)^2}\right]=\alpha\left[\frac{1}{1-e}-\frac{1}{1+e}\right]$$

$$\frac{L^2}{\cancel{2}\mu a}\frac{\cancel{4}e}{(1-e^2)^{\cancel{2}}}=\frac{\alpha\,\cancel{2e}}{\cancel{(1-e^2)}}$$

$$L=\sqrt{\mu a\alpha(1-e^2)}$$

recall now

$$\pi a^2\cancel{\sqrt{1-e^2}}=\frac{L}{2\mu}T=\frac{\sqrt{\mu a\alpha\cancel{(1-e^2)}}}{2\mu}T$$

$$\frac{2\pi a^2\sqrt{\mu}}{\sqrt{a\alpha}}=T$$

$$4\pi^2a^3\left(\frac{\mu}{\alpha}\right)=4\pi^2a^3\frac{\frac{m_1\cdot m_2}{m_1+m_2}}{G\,m_1m_2}=$$

$$\Rightarrow\boxed{\frac{4\pi^2a^3}{G(m_1+m_2)}=T^2}\ \boxed{\text{3rd Kepler's law}}$$

So 3rd Keppler's law is very powerful tool to find <u>total</u> <u>mass</u> of 2 body system.

This how we know mass of <u>Sun</u>.

$T = 1\text{year} \approx \pi \cdot 10^7 \text{ sec}$

$a \simeq 1\text{au} \approx 1.5 \cdot 10^{11} \text{ m}$ ← can be done with parallax observations

$G = 6.7 \cdot 10^{-11} \text{ Nm}^2/\text{kg}^2$

$$T^2 = \frac{4\pi^2 a^3}{G\underbrace{(m_\odot + m_\oplus)}_{M_T}}$$

$$M_T = \frac{4\pi^2 a^3}{G \cdot T^2} = \frac{4 \cdot (\pi)^2 \cdot \overbrace{(1.5 \cdot 10^{11})^3}^{\approx 3.6 \cdot 10^{33}}}{6.7 \cdot 10^{-11} \cdot (\pi \cdot 10^7)^2}$$

$$\cancel{\approx 20 \cdot 10^{33+11} = 2 \cdot 10^{45}}$$

$$\approx \frac{4 \cdot 3.6 \cdot 10^{33}}{6.7 \cdot 10^{3}} = \boxed{2 \cdot 10^{30} \text{ kg} = M_T}$$

$2 \approx$

But what is the mass of Earth? We cannot do it from $\odot \leftrightarrow \oplus$ system alone. Not enough precision in known numbers even if we can track Sun and Earth orbit to figurout $a_\odot$ and $a_\oplus$ and relate it to mass ratio.

Q: So what to do?: A: use Earth-Moon system

Even for Jupiter it is hard. but we can use moons of Jupiter to find its mass.

3rd Kepler's law revisited.

Interesting numerology :-)

$$T^2 = \frac{4\pi^2 a^3}{G(M_T)}$$

T : s → year

a : m → AU

$$T = T_y \cdot \frac{\text{sec in year}}{\text{year}}$$

$$a = a_{au} \cdot \frac{\text{m per au}}{1\,au}$$

$$\left[T_y \cdot (\pi \cdot 10^7)\frac{s}{y}\right]^2 = \frac{4\pi^2 \left(a_{au} \cdot 1.5 \cdot 10^{11} \frac{m}{au}\right)^3}{G\, M_T}$$

$$\frac{T_y^2}{a_{au}^3} = \frac{4\pi^2 (1.5 \cdot 10^{11})^3}{\pi^2 \cdot (10^7)^2 \cdot G \cdot M_T} =$$

num. coef. → $\beta \cdot M_\odot$

$$= \frac{4 \cdot \pi^2 (1.5 \cdot 10^{11})^3}{(\pi \cdot 10^7)^2 \cdot 6.7 \cdot 10^{-11} \beta\, 2 \cdot 10^{30}} =$$

// is it a coincidence? No.

$$\boxed{\frac{T_y^2}{a_{au}^3} \approx 1/\beta \quad , \quad \beta = \frac{M_T}{M_\odot}}$$

Q: if period of Jupiter is 11.86 years wha is ~~distan~~ it semi major axis?

$$a \approx \sqrt[3]{(12)^2} = \sqrt[3]{144} \approx 5.2\ au$$

which is the book value

Q: If sun were 25 times heavier, what would be Earth orbit period

A: ⟹ 1/5 of a year