Q: How to measure mass of Mercury and Venus which have no satellites.

A: you cannot (at least reliably) until you make such satellite your self. so mass of this planet were measured only a few decades ago. Before the were estimates which were wildly different.
\[ E = \frac{\mu U_P^2}{2} - \frac{\mu}{a(1-e)} \frac{L}{r} = \frac{\mu U_{a}^2}{2} - \frac{\mu}{a(1+e)} \frac{L}{r} \]

\[ L = \mu r_a U_a = \mu r_p U_P \]

\[ \Rightarrow \frac{r_a}{r_p} = \frac{U_p}{U_a} = \frac{\alpha (1+e)}{\alpha (1-e)} \]

\[ \frac{\mu U_P^2}{2} - \frac{L}{a(1-e)} = \frac{\mu U_{p}^2}{2} \left( \frac{1-e}{1+e} \right)^2 - \frac{L}{a(1+e)} \]

\[ U_P^2 = \frac{\alpha}{a} \left[ \frac{1}{1-e} - \frac{1}{1+e} \right] \]

\[ = \frac{\alpha}{a} \left[ \frac{2e}{1-e^2} \right] = \frac{\alpha}{\mu} \frac{(1+e)^2}{1-e^2} \]

\[ = \frac{\alpha}{\mu} \frac{1+e}{1-e} = \frac{GM_T}{a} \frac{1+e}{1-e} = U_P^2 \]

\[ V_a^2 = U_P^2 \left( \frac{1-e}{1+e} \right)^2 = \frac{GM_T}{a} \frac{1-e}{1+e} \]

\[ \text{Note:} \quad V_a^2 = U_P^2 \left( \frac{1-e}{1+e} \right)^2 = \frac{GM_T}{a} \frac{1-e}{1+e} \]
Halley comet revisited

See previous lecture

\[ e = 0.9673, \quad a = 18\text{au}, \quad b = 4.55\text{au} \]

\[ \sqrt{p} = \frac{GM\oplus}{a} \frac{1 + e}{1 - e} = \]

\[ = \sqrt{\frac{6.67 \times 10^{-11} \cdot 2 \times 10^{30}}{18 \cdot 1.5 \times 10^{10}}} \sqrt{\frac{1 + e}{1 - e}} = \]

\[ = 4.89 \quad \approx \sqrt{4.9 \times 10^{-7} \cdot \frac{1 + 0.9673}{1 - 0.9673}} = \]

\[ = 7.031 \cdot \sqrt{10^{10}} \approx 4.45 \]

\[ \approx 54.5 \text{ km/s} \]

\[ v_a = 906 \text{ m/s} = \sqrt{\frac{GM\oplus}{a} \frac{1 - e}{1 + e}} \]
What about Sun - Earth - Moon system?

Q: How orbit of moon looks like.

\[ \begin{align*}
M_\odot &= 2 \times 10^{30} \text{ kg} \\
M_\oplus &= 6 \times 10^{24} \text{ kg} \\
M_m &= 7.3 \times 10^{22} \\
T_\odot &= 1 \text{ au} = 1.5 \times 10^8 \text{ m} \\
T_{\text{EM}} &= 4 \times 10^8 \text{ m}
\end{align*} \]

\[ F_{\text{Sun-Moon}} = G \frac{M_\odot M_m}{T_\odot^2} = \]

\[ = 4.33 \times 10^{20} \text{ N} \]

\[ R_{\text{Sun Earth-moon}} = \frac{G M_\odot M_m}{M_\odot (T_{\text{EM}})^2} = \]

\[ = 1.8 \times 10^{20} \]

Earth speed on orbit \( e = 0 \)

\[ v_E = v_{\text{Moon}} = \sqrt{\frac{G M_\odot}{R}} = 30 \text{ km/s} \]

Moon speed around Earth = \( \sqrt{\frac{G M_\oplus}{T_{\text{EM}}}} = 10^3 \text{ m/s} = 1 \text{ km/s} \)

So Moon orbit with respect to Stars looks like...
In C.M system of Earth-Moon

What is the speed of Earth?

Recall \( r_1 = -\frac{M}{m_1} \cdot r \)

\( r_2 = \frac{M}{m_2} \cdot r \)

\[ \Rightarrow \quad \frac{u_1}{u_2} = \frac{r_1}{r_2} = \frac{m_2}{m_1} \]

\[ M = \frac{m_1 \cdot m_2}{m_1 + m_2} = \frac{M_M \cdot M_\oplus}{M_M + M_\oplus} \approx M_M \]

So if Moon speed

\( u_M \) then \( u_E = \frac{u_M}{\sqrt{m_1 + m_2}} \)

\[ u_E = \frac{M_M \cdot u_M}{\sqrt{m_1 + m_2}} = \frac{4.3 \cdot 10^{22}}{6 \cdot 10^{24}} \cdot 1000 \approx 12 \text{ m/s} \]
Almost always plane of the orbit is inclined:

\[ \text{C.M.} \] - appeared C.M. but for observed ellipse it should be in other point so there is a way to figure out inclination angle \( \theta \).

Note: move forward present as puzzle/Q

Simple example:

\[ M_1 = M_2 \quad e = 0 \quad \text{i.e. circular orbits} \]

In the inclined projection it will look like

But for ellipse c.m. must be in the focus! So observations contradict each other and need to be corrected.
Star mass measurements

We need 2 star bounded by gravitation, ideally we should be able to see both of them.

This put some constrains, on top of above orbital period should be ~ 1 year

Let's assume 2 stars with \( P = 1 \) year, what should be the distance between them.

\[
M_1 = M_\odot = M_2 \implies M_7 = 2M_\odot
\]

Recall derive

\[
1 \text{ year} = P^2 = \frac{a^3}{P} = \frac{a^3}{2} \implies \frac{M_1}{M_s}
\]

\[
a = \sqrt[3]{2} \approx 1.26 \text{ au}
\]

This quite close ~ distance of Sun - Earth
\[ a_1 = a \frac{M}{m_1}, \]
\[ a_2 = a \frac{M}{m_2} \]
\[ \Rightarrow a_1 + a_2 = a \left( \frac{M}{m_1} + \frac{M}{m_2} \right) = a \]

\[ \left( \frac{a_1}{a_2} \right) = \frac{m_2}{m_1}, \quad \text{observable} \]

Once we know \( a = a_1 + a_2 \) we can find total mass

\[ M_T = \frac{4\pi^2 a^3}{GP^2} \]

\[ M_T = m_1 + m_2 = m_1 + m_1 \frac{a_1}{a_2} = \]
\[ \Rightarrow \begin{align*}
  m_1 &= \frac{M_T}{1 + a_1/a_2} = \frac{a^2}{a_2 + a_1} M_T \\
  m_2 &= \frac{a_1}{a_1 + a_2} M_T
\end{align*} \]

So life sounds easy. But this only in the case of orbits in plane orthogonal to line of sight.
Once we know star masses, we can observe striking correlation between star mass and its luminosity (how much energy is emitted by star).

\[ \log \left( \frac{L}{L_\odot} \right) \] vs \[ \log \left( \frac{M}{M_\odot} \right) \]

The Mass-luminosity relationship