lecture 06

Well we would like to know

$\vec{r}(t)$ dependence.

For this we need extra info to use $\Rightarrow$

Energy conservation

$$E = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} + U(r) \qquad r = |\vec{r}_2 - \vec{r}_1|$$

C.M. ref. frame

$$= \frac{m_1 \left(-\frac{\mu}{m_1}\dot{\vec{r}}\right)^2}{2} + \frac{m_2 \left(\frac{\mu}{m_2}\dot{\vec{r}}\right)^2}{2} + U(r) = \qquad / |\dot{\vec{r}}| = v /$$

$$= \frac{1}{2}\left(m_1 \left(\frac{m_2}{m_1+m_2}\right)^2 + m_2 \left(\frac{m_1}{m_1+m_2}\right)^2\right) v^2 + U(r))$$

$$= \frac{1}{2} (\mu)^2 v^2 \Big(\underbrace{\frac{1}{m_1} + \frac{1}{m_2}}_{=\mu^{-1}}\Big) + U(r)$$

$$E = \frac{1}{2}\mu v^2 + U(r)$$

$$\vec{L} = \mu \vec{v} \times \vec{r} = \mu r^2 \dot{\theta}$$

Energy and angular momentum conservation

$$v^2 = (\vec{v}_\perp + \vec{v}_\parallel)\cdot(\vec{v}_\perp + \vec{v}_\parallel) =$$

$$= v_\parallel^2 + v_\perp^2 = (\dot{r})^2 + (r\dot{\theta})^2$$

$v_\parallel$: change of length for $\vec{r}$; $v_\perp$: change of direction for $\vec{r}$

$$E = \frac{1}{2}\mu(\dot{r})^2 + \frac{1}{2}\mu r^2(\dot{\theta})^2 + U(r)$$

recall $L = \mu r^2 \dot{\theta}$

$$\Rightarrow \quad E = \frac{1}{2}\mu(\dot{r})^2 + \frac{1}{2}\mu r^2\left(\frac{L}{\mu r^2}\right)^2 + U(r)$$

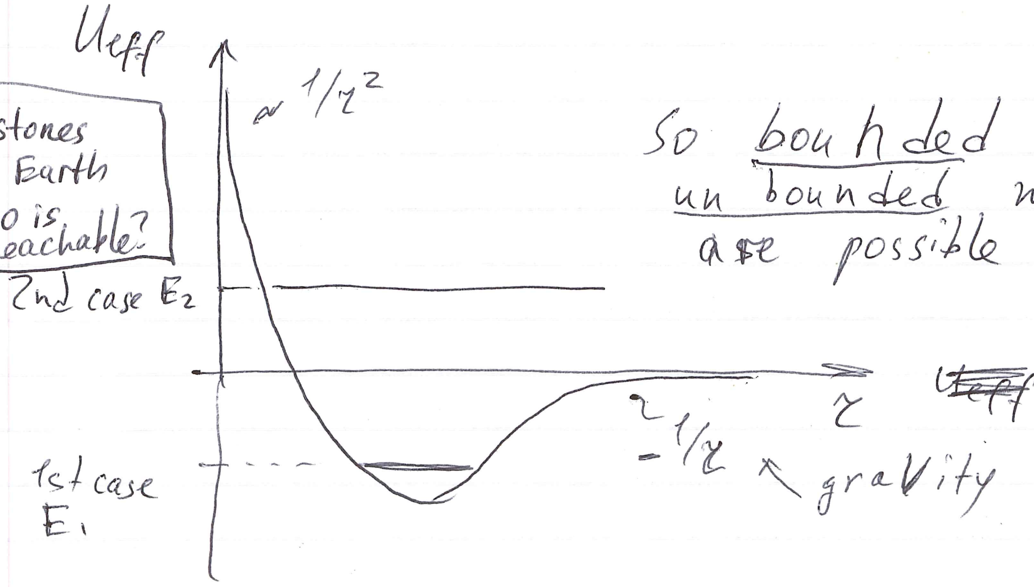
$$E = \frac{1}{2}\mu(\dot{r})^2 + \underbrace{\frac{L^2}{2\mu r^2} + U(r)}_{U_{eff}(r)}$$

$\frac{1}{2}\mu(\dot{r})^2$: look like kinetic energy

This already gives us something

So bounded and unbounded motions are possible

Q: why stones fall on Earth if r=0 is not reachable?



$$U(r) = -G\frac{m_1 m_2}{r} = -\frac{\alpha}{r}$$

$$\dot{r} = \sqrt{\frac{2}{\mu}(E - U_{eff})} = \frac{dr}{dt}$$

$$\frac{dr}{\sqrt{\frac{2}{\mu}(E-U_{eff})}} = dt = \frac{\mu r^2}{L}\, d\theta$$

$$d\theta = \frac{dr}{1}\,\frac{L}{\mu r^2\sqrt{\frac{2}{\mu}(E-U_{eff})}} = dr\,\frac{L/r^2}{\sqrt{2\mu(E-U_{eff})}}$$

(eq1) $$\int d\theta = \int \frac{dr\; L/r^2}{\sqrt{2\mu\left(E - \frac{L^2}{2\mu r^2} + \frac{\alpha}{r}\right)}} = \Big/\; \frac{1}{r} = x;\quad \frac{dr}{r^2} = -dx \;\Big/$$

$$= \int \frac{-L\,dx}{\sqrt{2\mu E - L^2x^2 - 2\mu\alpha x}}$$

HW: prove that eq1 => eq2

$= \Big/$ Note: $\frac{d}{dx}\arccos(x) = -\frac{1}{\sqrt{1-x^2}}$ $\Big/$

=>

(eq2) $$\theta = \arccos \frac{\frac{L}{r} - \frac{\mu\alpha}{L}}{\sqrt{2\mu E + \frac{\mu^2\alpha^2}{L^2}}} + \underbrace{const}_{\theta_0}$$

we rotate ref frame such that $\theta_0 = 0$

$$\cos\theta = \frac{\frac{L}{r} - \frac{\mu\alpha}{L}}{\sqrt{2\mu E + \frac{\mu^2\alpha^2}{L^2}}} =$$

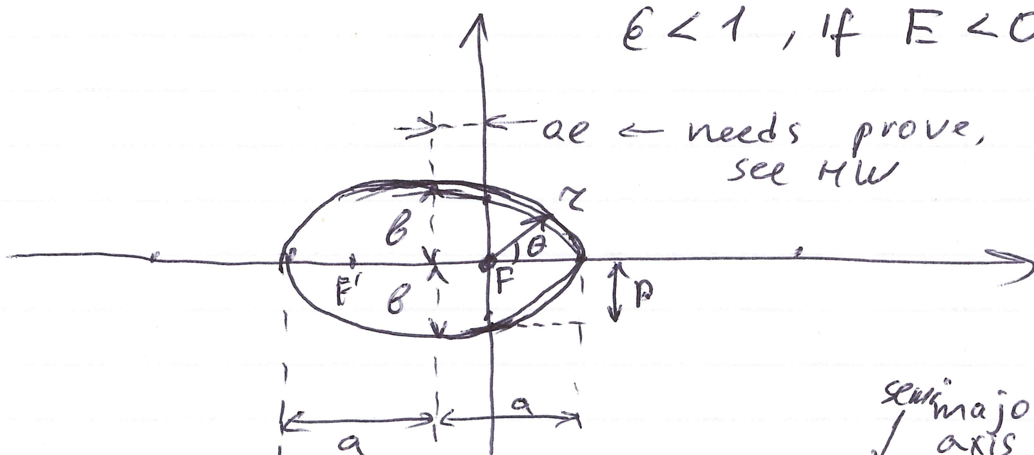
$$= \frac{\frac{L^2}{\mu\alpha}\frac{1}{r} - 1}{\sqrt{1 + \frac{2EL^2}{\mu\alpha^2}}}$$

eccentricity

$$p = \frac{L^2}{\mu\alpha^2} \text{ (const)} \; ; \; e = \sqrt{1 + \frac{2EL^2}{\mu\alpha^2}} \text{ (const)} \; ;$$

$$p/r = 1 + e\cdot\cos\theta$$

$e < 1$, if $E < 0$



semimajor axis

$$2a = \frac{p}{1+e} + \frac{p}{1-e} = \frac{2p}{1-e^2} \Rightarrow \boxed{a = \frac{p}{1-e^2} = \frac{\alpha}{2|E|}}$$

semiminor axis

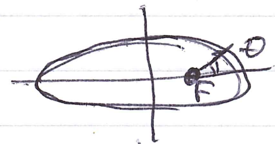
$$b = y_{max} = \{r\cdot\sin\theta\}_{max} = \frac{p}{\sqrt{1-e^2}} = \frac{L}{\sqrt{2\mu E}}$$

← (eq 3) HW

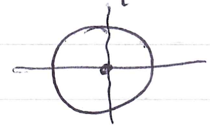
$$\boxed{b^2 = a^2(1-e^2)}$$

Orbits types

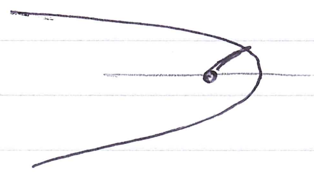
$e < 1$ ellipce



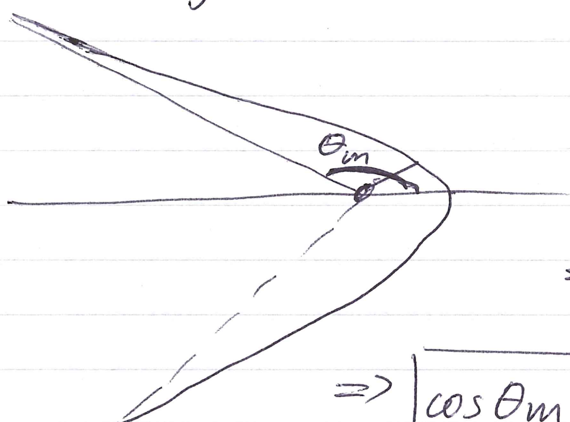
$e = 0 \rightarrow$ circle



$e = 1 \Rightarrow$ parabola



$e > 1 \Rightarrow$ hyperbola

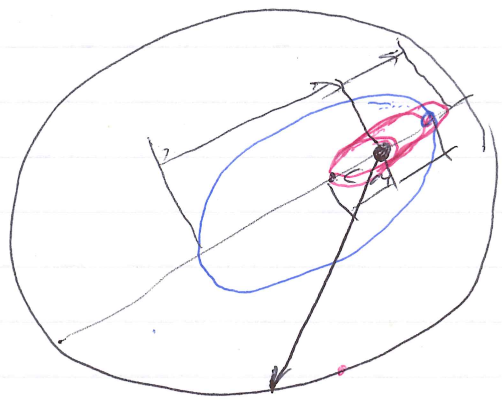


$$\frac{p}{r} = 1 + e\cos\theta$$

$r \rightarrow \infty$

$\Rightarrow$ ~~$\theta_m = \frac{1}{e}$~~

$$\Rightarrow \boxed{\cos\theta_m = -\frac{1}{e}}$$



$m_1 > m_2$

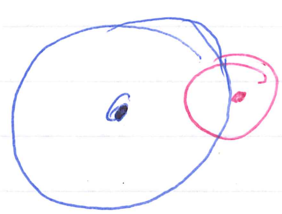
$r_1 = -\frac{\mu}{m_1} r$

$r_2 = \frac{\mu}{m_2} r$

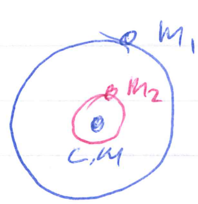
long ends on opposite sides of C.M.

if $e = 0 \Leftrightarrow$ circular orbits

is it possible to have situation like this?



No! Because they ~~orbit~~ move around the same C.M



Q: Which one is heavier?

A: $m_2 > m_1$