Derivation of Kepler's laws.
I.e., classical mechanics at work.

Kepler (1571 - 1630).

In 1609 first two laws

- 1- Orbits of the planets are ellipses, sun is at the foci of them & angular momentum conservation.

- 2- Line connecting a planet and the sun sweeps out equal areas at the same time interval.

10 years later

- 3- \( p^2 = a^3 \) average distance of a planet to the sun measured in au.

period of the orbit.
Classical mechanics

total angular momentum conservation

we have $N$ particles and no external force

so articles are acting only on themselves

\[
\vec{\mathbf{L}} = \sum_{i=1}^{N} m_i \vec{\mathbf{v}}_i \times \vec{\mathbf{r}}_i
\]

\[
\frac{d\vec{\mathbf{L}}}{dt} = \sum \left[ m_i \vec{\mathbf{a}}_i \times \vec{\mathbf{r}}_i + m_i \vec{\mathbf{v}}_i \times \vec{\mathbf{\dot{r}}}_i \right] = \sum m_i \vec{\mathbf{a}}_i \times \vec{\mathbf{r}}_i = \sum \vec{\mathbf{F}}_i \times \vec{\mathbf{r}}_i = \sum \left( \sum_{i \neq j} \vec{\mathbf{F}}_{ij} \times \vec{\mathbf{r}}_i \right)
\]

\[
\sum \sum_{i \neq j} \frac{|\vec{\mathbf{F}}_{ij}|}{|\vec{\mathbf{r}}_i - \vec{\mathbf{r}}_j|} \left( \vec{\mathbf{r}}_i - \vec{\mathbf{r}}_j \right) \times \vec{\mathbf{r}}_i = |\vec{\mathbf{F}}_{ij}| \left( \vec{\mathbf{r}}_i - \vec{\mathbf{r}}_j \right) \times \vec{\mathbf{r}}_i
\]

pairs of ions

\[
\Rightarrow \frac{d\mathbf{L}}{dt} = \text{const} \Rightarrow \mathbf{L} = \text{const}
\]
One particle around \textit{massive body}\hfill \textit{immobile}

\[ L_{\text{Total}} = m_1 \mathbf{r}_1 \times \mathbf{v}_1 = \text{const} \]

\[ L_{\text{total}} \cdot dt = m_1 (\mathbf{u}_1 \times \mathbf{r}_1) \, dt \]

\[ = m_1 (\mathbf{u}_1 \cdot dt) \times \mathbf{r}_1 \quad \Rightarrow \]

\[ |L_{\text{total}} \cdot dt| = m_1 \cdot (\mathbf{u}_1 \cdot \mathbf{r}_1 \cdot \sin \theta) \, dt \]

Area = \( \frac{1}{2} h \cdot r = \frac{1}{2} (\mathbf{u}_1 \cdot \sin \theta) \cdot r \)

So we proved \textit{2nd Kepler's law}
Now let's prove the 2nd Kepler's law in a more general case of a two-body system.

We introduce the center of mass position:

\[ \mathbf{r}_{CM} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} \]

Note:

\[ \mathbf{r}_2 = \mathbf{r}_1 + \mathbf{r} \]

The vector connecting 1 and 2 going from 1 to 2.

\[ \mathbf{r}_{CM} = \frac{m_1 \mathbf{r}_1 + m_2 (\mathbf{r}_1 + \mathbf{r})}{m_1 + m_2} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_1}{m_1 + m_2} = \mathbf{r}_1 + \frac{m_2}{m_1 + m_2} \mathbf{r}_2 \]

This proves that the C.M. sits on a line connecting 1 and 2.

If we move to the reference frame of C.M. than

\[ \mathbf{r}_{CM} = \mathbf{r}_1 - \mathbf{r}_{CM} = -\frac{m_2}{m_1 + m_2} \mathbf{r}_2 \]

\[ \mathbf{r}_{CM} = \mathbf{r}_1 - \frac{m_2 \mathbf{r}_2}{m_1 + m_2} \]

\[ M = \frac{m_2 m_1}{m_1 + m_2} \]

\[ \mu = \text{reduced mass} \]

\[ \mathbf{r}_{1 CM} = -\frac{M}{m_1} \mathbf{r}_2 \]

Similarly:

\[ \mathbf{r}_{2 CM} = +\frac{M}{m_2} \mathbf{r}_1 \]
For now on, we are always in the C.M. reference frame. So I dropped the C.M. subscript.

\[ \vec{L} = m_1 \vec{v}_1 \times \vec{r}_1 + m_2 \vec{v}_2 \times \vec{r}_2 = \]

\[ = m_1 \vec{v}_1 \times \left( -\frac{\vec{v}_1 \cdot \vec{r}_1}{m_1} \right) + m_2 \vec{v}_2 \times \left( \frac{\vec{v}_2 \cdot \vec{r}_2}{m_2} \right) = \]

\[ = M \left( -\vec{v}_1 \times \vec{r}_1 + \vec{v}_2 \times \vec{r}_2 \right) = \]

\[ = M \left( -\left( \frac{\vec{v}_1 \cdot \vec{r}_1}{m_1} \right) + \left( \frac{\vec{v}_2 \cdot \vec{r}_2}{m_2} \right) \right) \cdot \vec{r}_1 = \]

\[ = \mu \left( \frac{m_2}{m_1 + m_2} + \frac{m_1}{m_1 + m_2} \right) \cdot \vec{r}_2 \times \vec{r}_2 = \]

\[ = \mu \frac{\vec{v}_2 \times \vec{r}_2}{m_1 + m_2} = \vec{L}_{\text{total}} \]

\[ \vec{L} = \mu \left( \vec{v}_1 \times \vec{r}_1 + \vec{v}_2 \times \vec{r}_2 \right) = \mu \vec{v}_1 \times \vec{r}_1 + \mu \vec{v}_2 \times \vec{r}_2 \]

\[ = \mu (\vec{v}_1 \cdot \vec{r}_1) \cdot \vec{e}_L \]

unit vector along \( \vec{L} \)

which is \( \perp \) \( \vec{v}_1 \) and \( \vec{r}_1 \)
\[ \vec{r}_{(t+\Delta t)} - \vec{r}_{(t)} \]
\[ \Rightarrow \Delta \vec{r} = \vec{r}_{(t+\Delta t)} - \vec{r}_{(t)} \]
\[ \Rightarrow \Delta z = \Delta \vec{z} = \Delta z \cdot \Delta \theta \]
\[ \Rightarrow \hat{z} = \frac{\Delta z}{\Delta t} \Rightarrow \frac{\Delta \theta}{\Delta t} = \frac{\Delta \vec{z}}{\Delta t} \text{ for } \Delta t \to 0 \]

\[ |L| = \mu \cdot \Delta \theta \cdot \Delta z = \mu \cdot \vec{z} \cdot \Delta \theta = \mu \cdot z \cdot \Delta \theta \]
\[ = M \cdot 2 \frac{\text{dArea}}{\text{d}t} = \text{cons} \]

by previous prove

Same 2nd Kepler law,
i.e. it is equivalent to the angular momentum conservation

Note interesting fact it is true for any system without external forces. We do not care about the nature of the forces between the bodies - gravity, spring, coulomb force.

It is true for planets - sun, but also true for electron - atom, or 2 bodies connected by a thread.