Lecture 05

Derivation of Kepler's laws.

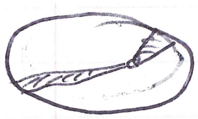
i.e. classical mechanics at work.

Kepler (1571 – 1630).

In 1609 first two laws

-1- Orbits of the planets are ellipses, sun is at the focus of them &

(angular momentum conservation) ⟺



-2- line connecting a planet and the sun sweeps out equal areas at the same time interval

(10 years later)

-3- $P^2 = a^3$

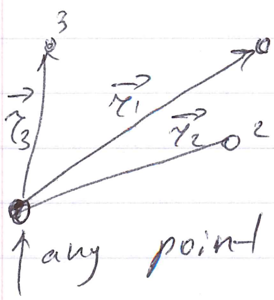
a: average distance of a planet to the sun measured in a.u.

P: period of ~~the~~ the orbit

Classical mechanics

total angular momentum conservation

we have N particles and <u>no external force</u>



so articles are acting only on themselves

$$\vec{L} = \sum_{i=1}^{N} m_i \vec{v}_i \times \vec{r}_i$$

$$\frac{d\vec{L}}{dt} = \sum \Big[m_i \dot{\vec{v}}_i \times \vec{r}_i + m_i \underbrace{\vec{v}_i \times \dot{\vec{r}}_i}_{\vec{v}_i \times \vec{v}_i = 0} \Big] =$$

$$= \sum m_i \vec{a}_i \times \vec{r} = \sum \vec{F}_i \times \vec{r}_i =$$

internal forces

$$= \sum_i \Big(\sum_{i \neq j} \vec{F}_{ij} \times \vec{r}_i \Big) =$$

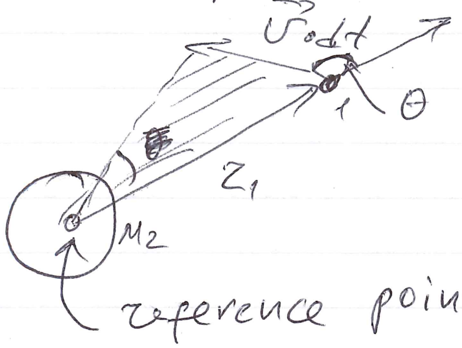
$$= \sum_i \sum_{i \neq j} \underbrace{|\vec{F}_{ij}| \cdot (\vec{r}_i - \vec{r}_j)}_{\text{Force between 2 bodies is along link}} \times \vec{r}_i =$$

$$= \sum_{\substack{\text{pairs} \\ \text{of } i \text{ and } j}} |F_{ij}| \cdot \underbrace{(\vec{r}_i - \vec{r}_j) \times \vec{r}_i}_{= -\vec{r}_j \times \vec{r}_i} - \underbrace{|F_{ij}| (\vec{r}_i - \vec{r}_j) \times \vec{r}_j}_{= \vec{r}_i \times \vec{r}_j}$$

$$= \sum_{\text{pairs}} |F_{ij}| \cdot \underbrace{\Big[\underbrace{-\vec{r}_j \times \vec{r}_i}_{= \vec{r}_i \times \vec{r}_j} - \vec{r}_i \times \vec{r}_j \Big]}_{= 0}$$

$$\Rightarrow \frac{d\vec{L}}{dt} = \text{const} \Rightarrow \boxed{\vec{L} = \text{const}} \, !$$

One particle around $\underbrace{\text{Massive}}^{\text{immobile}}$ body

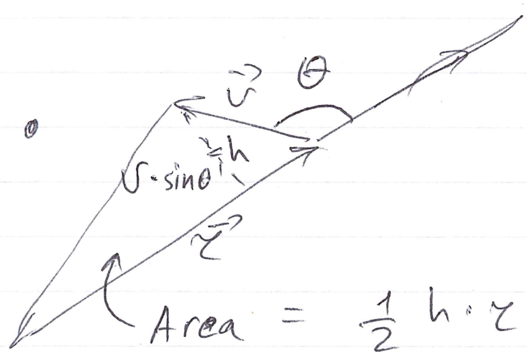


$$L_{Total} = m_1 \vec{v}_1 \times \vec{r}_1 = const$$

$$\vec{L}_{total} \cdot dt = m_1 (\vec{v}_1 \times \vec{r}_1) dt$$

$$= m_1 (\vec{v}_1 \cdot dt) \times \vec{r}_1 \quad \Rightarrow$$

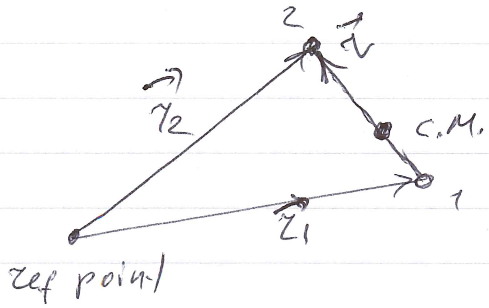
$$|\vec{L}_{total} \cdot dt| = m_1 \cdot \underbrace{(v_1 \cdot r_1 \cdot \sin\theta)}_{\substack{\parallel \\ const \\ \parallel \\ 2A \\ \parallel \\ h}} dt$$



$$\text{Area} = \frac{1}{2} h \cdot r = \frac{1}{2}(v \cdot \sin\theta) \cdot r$$

So we proved 2nd Kepler's law

Now let's prove 2nd Kepler's law in a more general case of 2 body system



We introduce center of Mass position

$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$$

note $\vec{r}_2 = \vec{r}_1 + \boxed{\vec{r}}$ ← vector connecting 1 and 2 going from 1 to 2

$$\vec{r}_{cm} = \frac{m_1 \cdot \vec{r}_1 + m_2\vec{r}_2}{m_1+m_2} = \frac{m_1 \cdot \vec{r}_1 + m_2(\vec{r}_1 + \vec{r})}{m_1+m_2} =$$

$$= \vec{r}_1 + \frac{m_2}{m_1+m_2}\vec{r}$$

This prove that the C.M sits on a line connecting 1 and 2

If we move to the reference frame of C.M. than

$$\vec{r}_{1cm} = \vec{r}_1 - \vec{r}_{cm} = -\frac{m_2}{m_1+m_2}\vec{r} =$$

$$= - \frac{m_2 \cdot m_1}{(m_1+m_2)\cdot m_1} = -$$

$$\boxed{\mu = \frac{m_2 m_1}{m_1+m_2}}$$

μ - reduced mass

$$\boxed{r_{1cm} = -\frac{\mu}{m_1}\vec{r}}$$, similarly $$\boxed{r_{2cm} = +\frac{\mu}{m_2}\vec{r}}$$

For now on, we are always in the c.M. reference frame. So I dropped the c.M. subscript

(P5)

$$\vec{L} = m_1 \vec{v_1} \times \vec{r_1} + m_2 \vec{v_2} \times \vec{r_2} =$$

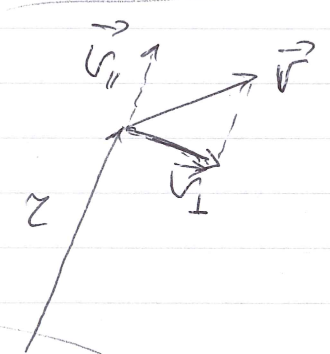
$$= m_1 \vec{v_1} \times \left(-\frac{\vec{r}\mu}{m_1}\right) + m_2 \vec{v_2} \times \left(\vec{r}\frac{\mu}{m_2}\right) =$$

$$= \mu\left(-\vec{v_1} \times \vec{r} + \vec{v_2} \times \vec{r}\right) =$$

$$= \mu\left(-\left(\frac{\mu}{m_1}\dot{\vec{r}}\right) + \left(\frac{\mu}{m_2}\dot{\vec{r}}\right)\right) \times \vec{r} =$$

$$= \mu\underbrace{\left(\frac{m_2}{m_1+m_2} + \frac{m_1}{m_1+m_2}\right)}_{=1}\dot{\vec{r}} \times \vec{r} =$$

$$= \mu\,\dot{\vec{r}} \times \vec{r} = \boxed{\mu\,\vec{v} \times \vec{r} = \vec{L}_{total}}$$

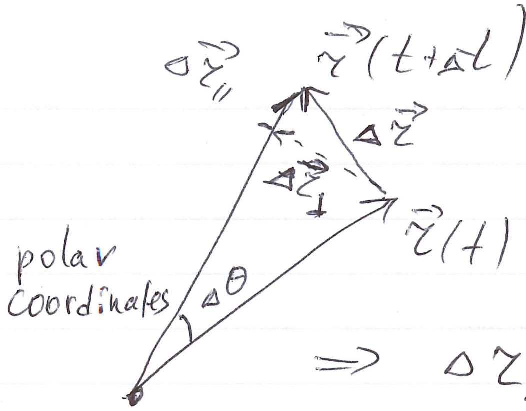


$$\vec{v} = \vec{v}_{\perp} + \vec{v}_{\parallel}$$

$$= \mu\left(\vec{v}_{\perp} + \vec{v}_{\parallel}\right) \times \vec{r} = \mu\,\vec{v}_{\perp} \times \vec{r} + \mu\,\underbrace{\vec{v}_{\parallel} \times \vec{r}}_{0\ (\vec{v}_{\parallel} \parallel \vec{r})}$$

$$= \mu(v_{\perp} \cdot r) \cdot \vec{e}_L$$

unit vector along $\vec{L}$ which is $\perp$ to $v_{\perp}$ and r



$$\Delta \vec{r} = \Delta \vec{r}_{\perp} + \Delta \vec{r}_{\parallel}$$

$$\Rightarrow \Delta r_{\perp} = r \cdot \Delta\theta$$

$$\Rightarrow v_{\perp} = \frac{\Delta r_{\perp}}{\Delta t} = \frac{r \Delta\theta}{\Delta t} \xrightarrow[\Delta t \to 0]{} r\dot{\theta}$$

$$\boxed{v_{\perp} = r\dot{\theta}}$$

$$\Rightarrow |L| = \mu\, r \cdot \dot{\theta} \cdot r = \mu r^2 \dot{\theta} =$$

$$= \mu \cdot 2 \frac{d\,\text{Area}'}{dt} = \text{const}$$

by previous prove

Same 2nd Keppler law.
i.e. it is equivalent to the angular
~~Note~~ momentum conservation

Note interesting fact it is true for any system without external forces. We do not care about the nature of the forces between the bodies: gravity, spring, ~~Electric~~ Coulomb force.

It is true for planets - sun, but also true for electron - atom, or 2 bodies connected by a thread