Discrete Fourier Transform and filters

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Lecture 26

Notes

DFT vs. Matlab FFT

DFT

$$y_k = \frac{1}{N}\sum_{n=0}^{N-1} c_n \exp(i\frac{2\pi(k-1)n}{N}) \quad \text{inverse Fourier transform}$$

$$c_n = \sum_{k=1}^{N} y_k \exp(-i\frac{2\pi(k-1)n}{N}) \quad \text{Fourier transform}$$

$$n = 0, 1, 2, \cdots, N-1$$

Matlab FFT

$$y_k = \frac{1}{N}\sum_{n=1}^{N} c_n \exp(i\frac{2\pi(k-1)(n-1)}{N}) \quad \text{inverse Fourier transform}$$

$$c_n = \sum_{k=1}^{N} y_k \exp(-i\frac{2\pi(k-1)(n-1)}{N}) \quad \text{Fourier transform}$$

$$n = 1, 2, \cdots, N$$

So do DFT $\rightarrow$ Matlab FFT is equivalent of $n \rightarrow n+1$ and vice versa

Notes

Warning about notation

c_0 has a special meaning. It is the 0 frequency (i.e., DC) amplitude of the signal. Thus, I will always use the DFT notation unless mentioned otherwise.

People often denote the forward Fourier transform as $\mathcal{F}$

$$Y = \mathcal{F}y$$

So $Y = (Y0, Y1, Y2, \ldots, Y_{N-1}) = (c_0, c_1, c_2, \ldots, c_{N-1})$ is the spectrum of the time domain signal y

Inverse Fourier transform is denoted as $\mathcal{F}^{-1}$

$$y = \mathcal{F}^{-1}Y$$

Instead of using c_n coefficients, we refer in this notation to Y_n

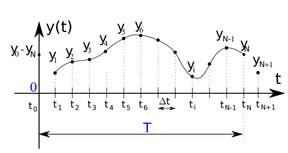
Notes

Sampling rate and important physics relationship

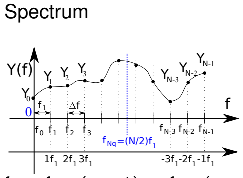
For DFT we need to have equidistant points and the signal repeating itself. We consider signals which start at time 0 and take N points over the period time T, thus, $y_k = y_{k+N}$. To deduce the time of a data point, we just multiply its index by the time spacing $\Delta t = T/N$. I.e., y_i is taken at time $t_i = i\Delta t = i/f_s$

The sampling rate f_s is defined as $f_s = 1/\Delta t = f_1 N$, and $f_1 = f_s/N = 1/T$ is the frequency spacing in the spectrum, sometimes it is referred as the resolution bandwidth (RBW).

Time series



Spectrum



In Matlab `fft`, Y_n has the frequency $f_n = f_1 \times (n-1) = f_s \times (n-1)/N$.

Notes

Nyquist frequency

If we take N data points with the sampling rate f_s, what is the maximum frequency which we can expect to see in our spectrum?
Naively, we can say $(N-1) \times f_1 \approx f_s$, since in the DFT spectrum all points are separated by the fundamental frequency $f_1 = 1/T = f_s/N$
However, recall that

$$Y_n = c_n = \sum_{k=1}^{N} y_k \exp(-i\frac{2\pi(k-1)n}{N})$$

Thus, $Y_{N-n} = Y_{-n}$, i.e., the higher half of the vector Y contains negative frequency. So at most, we can hope to obtain a spectrum with the highest frequency smaller than

Nyquist frequency

$$F_{Nq} = f_1\frac{N}{2} = \frac{f_s}{2}$$

Notes

Nyquist criteria

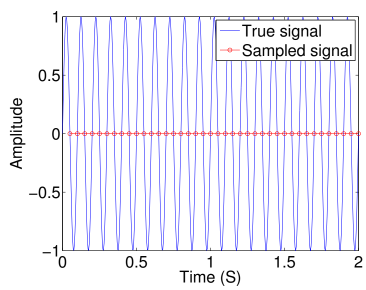
$$f_s > 2f_{signal}$$

You must sample your signal twice faster than the highest frequency component of it. I.e., the Nyquist frequency of your sample should be > than the highest signal frequency.

Notes

Aliasing: wrong/slow sampling frequency

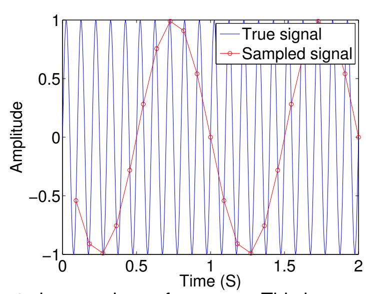
Sampling with
$f_s = 2f_{signal}$
i.e.
$f_{Nq} = f_{signal}$
Sampled signal appeared to be DC



Notes

Aliasing: too slow sampling frequency - reflection

Under sampling
$f_s = 1.1f_{signal}$
The sampled signal seems to have a lower frequency.

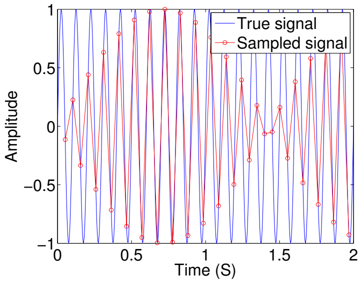


The sampled signal appears to have a slower frequency. This is case of the reflection/folding where the signal frequency is slightly higher than the sampling frequency.
$f_{apparent\ signal} = (f_{signal} - 2f_{Nq}) \approx f_{signal} - f_s$

Notes

Aliasing: too slow sampling frequency - ghosts

Under sampling
$f_s = 1.93 f_{signal}$
The sampled signal looks very different.



DFT filters

Once you get a signal, you can filter the unwanted frequencies out of it. The recipe is the following

- sample the signal
- calculate DFT (use Matlab `fft`)
- have a look at the spectrum and decide which frequencies are unwanted
- apply a filter which attenuate unwanted frequencies amplitudes
  - If you attenuate the component of the frequency f by g_f, you need to attenuate the component at $-f$ by g_f^*. Otherwise, the inverse Fourier transform will have non zero imaginary part.
- calculate inverse DFT (`ifft`) of the filtered spectrum
- repeat if needed

Applications

- Noise reduction
- Compression

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