

Fourier transform

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Lecture 25

Fourier series

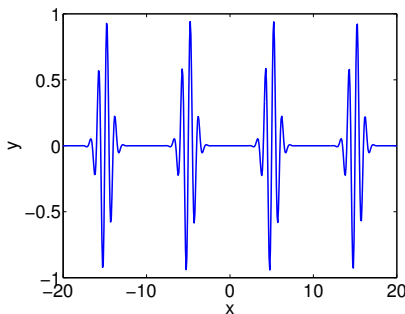
Any periodic single value function with a finite number of discontinuities, and for which $\int_0^T |f(t)| dt$ is finite, can be presented as

$$y(t) = \frac{a_0}{2} + \sum_1^{\infty} (a_n \cos(n\omega_1 t) + b_n \sin(n\omega_1 t))$$

T is the period, i.e., $y(t) = y(t + T)$
 $\omega_1 = 2\pi/T$ is the fundamental frequency

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \frac{2}{T} \int_0^T dt \begin{pmatrix} \cos(n\omega_1 t) \\ \sin(n\omega_1 t) \end{pmatrix} y(t)$$

At a discontinuity, the series approaches the mid point

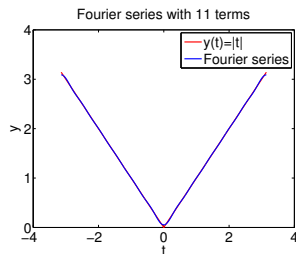
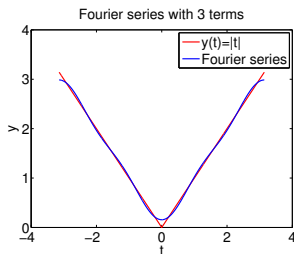
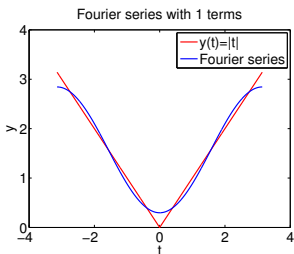
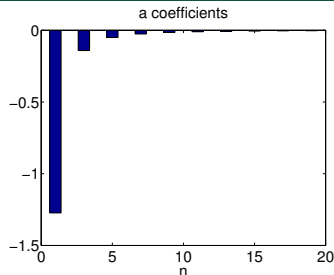


Fourier series example: $|t|$

$$y(t) = |t|, \quad -\pi < t < \pi$$

Since the function is even all $b_n = 0$

$$\begin{cases} a_0 = \pi, \\ a_n = 0, & n \text{ is even} \\ a_n = -\frac{4}{\pi n^2}, & n \text{ is odd} \end{cases}$$

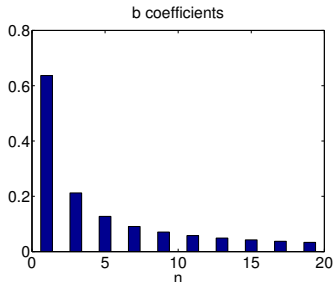


Fourier series example: step function

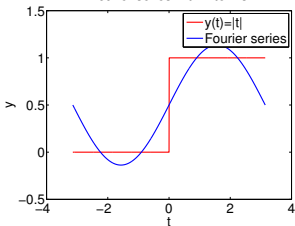
$$\begin{cases} 0, & -\pi < x < 0, \\ 1, & 0 < x < \pi \end{cases}$$

Since the function is odd all $a_n = 0$ except $a_0 = 1$

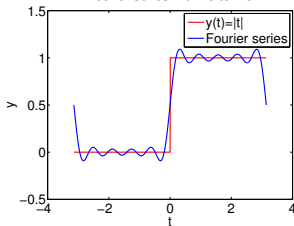
$$\begin{cases} b_n = 0, & n \text{ is even} \\ b_n = \frac{2}{\pi n}, & n \text{ is odd} \end{cases}$$



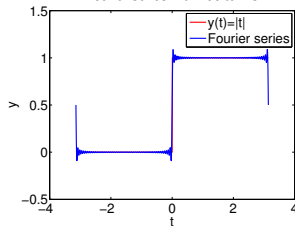
Fourier series with 1 terms



Fourier series with 10 terms



Fourier series with 100 terms



Complex representation

Recall that

$$\exp(i\omega t) = \cos(\omega t) + i \sin(\omega t)$$

It can be shown that

$$y(t) = \sum_{n=-\infty}^{\infty} c_n \exp(in\omega_1 t)$$
$$c_n = \frac{1}{T} \int_0^T y(t) \exp(-in\omega_1 t) dt$$

$$a_n = c_n + c_{-n}$$

$$b_n = i(c_n - c_{-n})$$

What to do if function is not periodic?

- $T \rightarrow \infty$
- $\sum \rightarrow \int$
- discrete spectrum \rightarrow continuous spectrum
 - $c_n \rightarrow c_\omega$

$$y(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c_\omega \exp(i\omega t) d\omega$$
$$c_\omega = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y(t) \exp(-i\omega t) dt$$

Above requires that $\int_{-\infty}^{\infty} dt y(t)$ exists and is finite.

Note that c_ω has the extra $\sqrt{2\pi}$ when compared to c_n , and T is gone.

Discrete Fourier transform (DFT)

In reality, **we cannot have**

- infinitely large interval
- infinite amount of points to calculate true integral

Assuming that $y(t)$ has a period T and we took N **equidistant** points

$$\Delta t = \frac{T}{N} \text{ samples spacing, } f_s = \frac{1}{\Delta t} \text{ sampling rate}$$

$$f_1 = \frac{1}{T} = \frac{1}{N\Delta t} \text{ smallest observed frequency,}$$

also resolution bandwidth

$$t_k = \Delta t \times (k - 1)$$

$$y(t_{k+N}) = y(t_k) \text{ periodicity condition}$$

$$y_k = y(t_k) \text{ shortcut notation}$$

$$y_1, y_2, y_3, \dots, y_N \quad \text{data set}$$

We replace the integral in the Fourier series with the sum

$$y_k = \frac{1}{N} \sum_{n=0}^{N-1} c_n \exp(i \frac{2\pi(k-1)n}{N}) \quad \text{inverse Fourier transform}$$

$$c_n = \sum_{k=1}^N y_k \exp(-i \frac{2\pi(k-1)n}{N}) \quad \text{Fourier transform}$$

$$n = 0, 1, 2, \dots, N-1$$

Confusion keeps increasing: where are the negative coefficients c_{-n} ?
 In DFT, they moved to the right end of the c_n vector :

$$c_{-n} = c_{N-n}$$

Fast Fourier transform (FFT)

Fast numerical realization of DFT is FFT. This is just the smart way to do DFT. Matlab has one built in

- y is a matlab vector of data points (y_k)
- $c = \text{fft}(y)$ Fourier transform
- $y = \text{ifft}(c)$ inverse Fourier transform

Notice that `fft` does not normalize by N . So to get Fourier series c_n , you need to calculate $\text{fft}(y) / N$.

However $y = \text{ifft}(\text{fft}(y))$

Notice one more point of confusion: Matlab does not have index=0, so actual $c_n = c_{\text{matlab fft}(n-1)}$, so $c_0 = c_{\text{matlab fft}(1)}$