

System of linear algebraic equations

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Lecture 21

Mobile problem

Someone provided us with 6 weights and 3 rods. We need to calculate the positions of suspension points to have a balanced system.

If the system is in equilibrium, torque must be zero at every pivot point

$$w_1 x_1 - (w_2 + w_3 + w_4 + w_5 + w_6) x_2 = 0$$

$$w_3 x_3 - (w_4 + w_5 + w_6) x_4 = 0$$

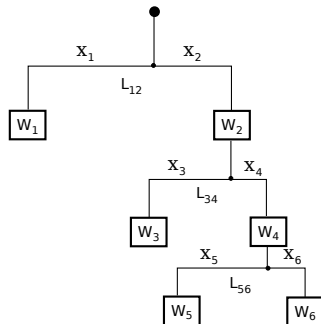
$$w_5 x_5 - w_6 x_6 = 0$$

We need 3 more equations. Let's constrain the length of every rod

$$x_1 + x_2 = L_{12}$$

$$x_3 + x_4 = L_{34}$$

$$x_5 + x_6 = L_{56}$$



Mobile problem continued

Let's define $w_{26} = w_2 + w_3 + w_4 + w_5 + w_6$ and $w_{46} = w_4 + w_5 + w_6$

$$w_1 x_1 - w_{26} x_2 = 0$$

$$w_3 x_3 - w_{46} x_4 = 0$$

$$w_5 x_5 - w_6 x_6 = 0$$

$$x_1 + x_2 = L_{12}$$

$$x_3 + x_4 = L_{34}$$

$$x_5 + x_6 = L_{56}$$

$$\sum_j A_{ij} x_j = B_i \rightarrow \mathbf{Ax} = \mathbf{B}$$

Matlab has a lot of built-in functions to solve problems in this form

$$\begin{pmatrix} w_1 & -w_{26} & 0 & 0 & 0 & 0 \\ 0 & 0 & w_3 & -w_{46} & 0 & 0 \\ 0 & 0 & 0 & 0 & w_5 & -w_6 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ L_{12} \\ L_{34} \\ L_{56} \end{pmatrix}$$

The inverse matrix method

$$\mathbf{Ax} = \mathbf{B}$$

$$\mathbf{A}^{-1}\mathbf{Ax} = \mathbf{x} = \mathbf{A}^{-1}\mathbf{B}$$

Analytical solution

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{B}, \quad \text{only if } \det(\mathbf{A}) \neq 0$$

Matlab's straight forward implementation (not the fastest)

$$\mathbf{x} = \text{inv}(\mathbf{A}) * \mathbf{B}$$

Matlab's faster way with the left division operator (recommended)

$$\mathbf{x} = \mathbf{A} \setminus \mathbf{B}$$

Recall the mobile problem

If $w_1 = 20$, $w_2 = 5$, $w_3 = 3$, $w_4 = 7$, $w_5 = 2$, $w_6 = 3$, $L_{12} = 2$, $L_{34} = 1$, and $L_{56} = 3$, then $w_{26} = 20$ and $w_{46} = 12$.

$$\begin{pmatrix} 20 & -20 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & -12 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -3 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 1 \\ 3 \end{pmatrix}$$

Matlab mobile solution

```
A=[ ...  
20, -20, 0, 0, 0, 0; ...  
0, 0, 3, -12, 0, 0; ...  
0, 0, 0, 0, 2, -3; ...  
1, 1, 0, 0, 0, 0; ...  
0, 0, 1, 1, 0, 0; ...  
0, 0, 0, 0, 1, 1; ...  
]  
B= [ 0; 0; 0; 2; 1; 3 ]  
% 1st method  
x=inv(A)*B  
% 2nd method  
x=A\B
```

```
x =  
1.0000  
1.0000  
0.8000  
0.2000  
1.8000  
1.2000
```

Check

```
>> A*x-B  
1.0e-15 *  
0  
0  
0  
0  
0.2220  
0
```

To do or not to do the inverse matrix calculation

Solutions based on the inverse matrix calculation involve extra steps (unnecessary for solution) and, thus, **are slower**

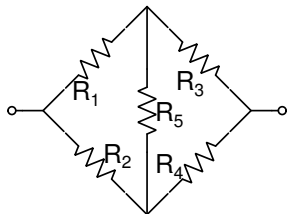
```
>> A=rand(4000);  
>> B=rand(4000,1);  
>> tic; x=inv(A)*B; toc  
Elapsed time is 54.831124 seconds.  
>> tic; x=A\B; toc  
Elapsed time is 19.822778 seconds.
```

However, it is handy to calculate the inverse matrix in advance if you solve $\mathbf{Ax} = \mathbf{B}$ for different \mathbf{B} with the same \mathbf{A} .

```
>> tic; Ainv=inv(A); toc  
Elapsed time is 58.304244 seconds.  
>> B1=rand(4000,1); tic; x1=Ainv*B1; toc  
Elapsed time is 0.048547 seconds.  
>> B2=rand(4000,1); tic; x2=Ainv*B2; toc  
Elapsed time is 0.048315 seconds.
```

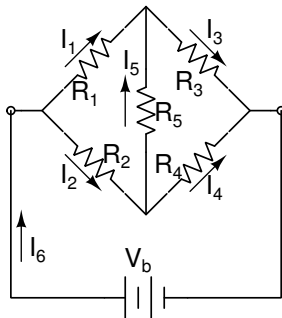
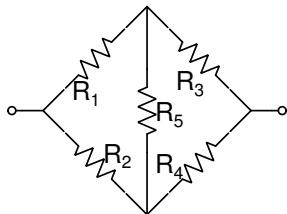
Wheatstone bridge problem

Find the equivalent resistance of the following combination of resistors.



Wheatstone bridge problem

Find the equivalent resistance of the following combination of resistors.



$$R_{eq} = \frac{V_b}{I_6}$$

Wheatstone bridge problem solution

```
%% Wheatstone bridge calculations
R1=1e3; R2=1e3; R3=2e3; R4=2e3; R5=10e3;
Vb=9;
A=[
    -1, -1, 0, 0, 0, 1; % I1+I2=I6 eq1
    1, 0, -1, 0, 1, 0; % I1+I5=I3 eq2
    0, 1, 0, -1, -1, 0; % I4+I5=I2 eq3
    % 0, 0, 1, 1, 0, -1; % I3+I4=I6 eq4
    % above would make a linear combination
    % of the following eq1+eq2=-(e3+eq4)
    0, 0, R3, -R4, R5, 0; % R3*I3+R5*I5=R4*I4 eq4a
    R1, 0, R3, 0, 0, 0; % R1*I1+R3*I3=Vb eq5
    -R1, R2, 0, 0, R5, 0 % R2*I2+R5*I5=R1*I1 eq6
]
B=[0; 0; 0; 0; Vb; 0];

% Find currents
I=A\B

% equivalent resistance of the Wheatstone bridge
Req=Vb/I(6)
```