Ordinary Differential equations

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Lecture 19

ODE definitions

An ordinary equation of order n has the following form

$$y^{(n)} = f(x, y, y', y'', \cdots, y^{(n-1)})$$

x independent variable

$y^{(i)} = \frac{\partial^i y}{\partial x^i}$, the i_{th} derivative of $y(x)$

f the force term

First order ODE example

Example

the acceleration of a body is the first derivative of velocity with respect to the time and equals to the force divided by mass

$$a(t) = \frac{dv}{dt} = v'(t) = \frac{F}{m}$$

$$\begin{aligned} t &\to x \text{ independent variable} \\ v &\to y \\ F/m &\to f \end{aligned}$$

And we obtain the canonical form

$$y^{(1)} = f(x, y)$$

for the first order ODE

n_{th} order ODE transformation to the system of first order ODEs

$$y^{(n)} = f(x, y, y', y'', \cdots, y^{(n-1)})$$

We define the following variables

$$y_1 = y, y_2 = y', y_3 = y'', \cdots, y_n = y^{(n-1)}$$

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \\ \vdots \\ y_{n-1}' \\ y_n' \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{n-1} \\ f_n \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ y_4 \\ \vdots \\ y_n \\ f(x, y_1, y_2, y_3, \cdots y_n) \end{pmatrix}$$

We can rewrite n_{th} order ODE as a system of first order ODEs

$$\vec{y}' = \vec{f}(x, \vec{y})$$

Cauchy boundary conditions

$$\vec{y}' = \vec{f}(x, \vec{y})$$

This is the system of n equations and thus requires n constraints.

With Cauchy boundary conditions, we specify $\vec{y}(x_0) = \vec{y}_0$ i.e. initial conditions

$$\begin{pmatrix} y_1(x_0) \\ y_2(x_0) \\ y_3(x_0) \\ \vdots \\ y_n(x_0) \end{pmatrix} = \begin{pmatrix} y_{1_0} \\ y_{2_0} \\ y_{3_0} \\ \vdots \\ y_{n_0} \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \\ y_0'' \\ \vdots \\ y_0^{(n-1)} \end{pmatrix}$$

Problem example

If acceleration of the particle is given, then we find the position of the particle as a function of time by solving

$$x''(t) = a$$

First, we need to convert it to the canonical form: the system of the first order ODEs.

$$\begin{aligned} t &\to x \text{ time as independent variable} \\ x &\to y \to y_1 \text{ particle position} \\ v &\to y' \to y_2 \text{ velocity} \\ a &\to f \text{ acceleration as a force term} \end{aligned}$$

so

$$x'' = a \to y'' = f \to \vec{y}' = \vec{f}(x, \vec{y}) \to \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2 \\ f \end{pmatrix}$$

We also need to provide the initial conditions:
position $x_0 \to y_{1_0}$ and velocity $v_0 \to y_{2_0}$

Euler's method

Let's, for simplicity, consider a simple first order ODE (notice the lack of the vector notation)

$$y' = f(x, y)$$

There is an exact way to write the solution

$$y(x_f) = y(x_0) + \int_{x_0}^{x_f} f(x, y)dx$$

The problem is that $f(x, y)$ depends on y itself. However, on a small interval $[x, x + h]$, we can assume that $f(x, y)$ is constant.
Then, we can use the familiar box integration formula. In application to the ODE, this is called the Euler's method.

$$y(x + h) = y(x) + \int_{x}^{x+h} f(x, y(x))dx \approx y(x) + f(x, y(x))h$$

Euler's method continued

$$y(x+h) = y(x) + f(x,y)h$$

All we need is to split our interval into a bunch of steps of the size h and leap frog from the first x_0 to the next one $x_0 + h$, then to the $x_0 + 2h$, and so on.

Now, we can make an easy transformation to the vector case (i.e. the n_{th} order ODE)

$$\vec{y}(x+h) = \vec{y}(x) + \vec{f}(x,y)h$$

Similarly to the boxes integration method, which is inferior in comparison to more advance methods, for example, the trapezoidal and Simpson's, the Euler's method is very imprecise for a given h and there are better ways.

Stability issues for the numerical solution

Let's have a look at the first order ODE

$$y' = 3y - 4e^{-x}$$

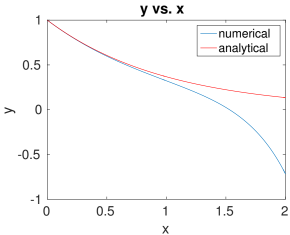
It has the following analytical solution

$$y = Ce^{3x} + e^{-x}$$

If the initial condition $y(0) = 1$, then the solution is

$$y(x) = e^{-x}$$

The `ode_unstable_example.m` script compares the numerical and the analytical solutions



It is clear that the numerical solution diverges from the analytical solution. The problem is in the round off errors. From a computer point of view, $y(0) = 1 + \delta$. Thus, $C \neq 0$ and the numerical solution diverges.
Do not trust the numerical solutions (regardless of a method) without proper consideration!