

# Optimization problem

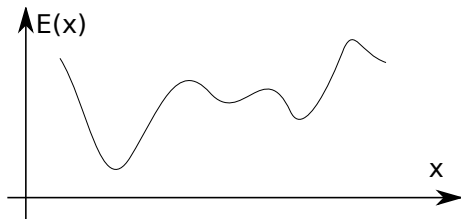
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Lecture 15

# Introduction to optimization



Find  $\vec{x}$  that minimizes  $E(\vec{x})$  subject to  $g(\vec{x}) = 0, h(\vec{x}) \leq 0$

$\vec{x}$  design variables

$E(\vec{x})$  merit or objective or fitness or energy function

$g(\vec{x})$  and  $h(\vec{x})$  constrains

Easy to see that maximization problem is the same as minimization once  $E(\vec{x}) \rightarrow -E(\vec{x})$ .

In general, there is no guaranteed way (i.e., algorithm) to find the **global minimum** point at finite time in a general case.

# Analytical solution of the 1D case

If we have the 1D case and  $E(x)$  has the analytical derivative, the optimization problem can be restated as

Find  $x$  such that  $f(x) = 0$   
where  $f(x) = dE/dx$

We already know how to find the solution of  $f(x) = 0$ , so the rest is easy. Note that we will find a **local** minimum or maximum.

# Example: the maximum of a black body radiation spectrum

According to Planck's law energy density per of black body radiation

$$I(\lambda, T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda kT}} - 1}$$

where

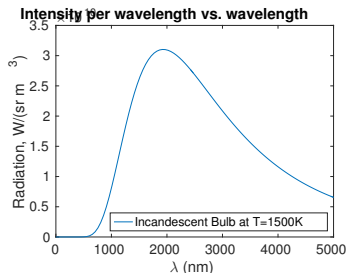
$h$  is the Planck constant  $6.626 \times 10^{-34} \text{ J}\cdot\text{s}$ ,

$c$  is the speed of light  $2.998 \times 10^8 \text{ m/s}$ ,

$k$  is the Boltzmann constant  $1.380 \times 10^{-23} \text{ J/K}$ ,

$T$  is the body temperature in K,

$\lambda$  is the wavelength in m



# 1D minimization with the Matlab's built-in: `fminbnd`

```
function I_lambda=black_body_radiation(lambda,T)
% black body radiation spectrum
% lambda – wavelength of EM wave
% T – temperature of a black body
h=6.626e-34; % the Plank constant
c=2.998e8; % the speed of light
k=1.380e-23; % the Boltzmann constant

I_lambda = 2*h*c^2 ./ (lambda.^5) ./ (exp(h*c./(lambda*k*T)) - 1);
end
```

First, we flip/negate the function since our algorithm is suited for a minimum search and set the T close to Sun temperature

```
T=5778;
f = @(x) - black_body_radiation(x,T);
```

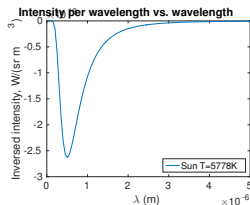
Finally, we find the minimum location

```
fminbnd(f,1e-9,2e-6,optimset('ToI',1e-12))
```

```
ans = 5.0176e-07
```

% i.e., the maximum of Sun radiation is at 502 nm

Next, we plot it to find a bracket



# Golden section search algorithm

If you have an initial bracket for solution i.e. found  $a, b$  points such that there is a point  $x_p$  satisfying  $a < x_p < b$  and  $E(x_p) < \min(E(a), E(b))$ .

- 1 Calculate  $h = (b - a)$
- 2 Assign new probe points  $x_1 = a + R * h$  and  $x_2 = b - R * h$
- 3  $E_1 = E(x_1), E_2 = E(x_2), E_a = E(a), E_b = E(b)$
- 4 Note that for small enough  $h$ :  $E(x_1) < E(a)$  and  $E(x_2) < E(b)$
- 5 Shrink/update the bracket
  - if  $E_1 < E_2$  then  $b = x_2, E_b = E_2$  else  $a = x_1, E_a = E_1$
- 6 if  $h < \varepsilon_x$  then stop otherwise do steps below
- 7 **With the proper  $R$** , we can reuse one of the old points; either  $x_1, E_1$  or  $x_2, E_2$  Thus, **we reduce the calculation time**
  - if  $E_1 < E_2$   
then  $x_2 = x_1, E_2 = E_1, x_1 = a + R * h, E_1 = E(x_1)$   
else  $x_1 = x_2, E_1 = E_2, x_2 = b - R * h, E_2 = E(x_2)$
- 8 Go to step 5

The  $R$  is given by the golden section  $R = \frac{3-\sqrt{5}}{2} \approx 0.38197$

# Derivation of the $R$ value

at the first step we have

$$x_1 = a + R * h$$

$$x_2 = b - R * h$$

If  $E(x_1) < E(x_2)$ , then  $a' = a$  and  $b' = x_2$  then, to find the next bracket, we evaluate  $x'_1$  and  $x'_2$

$$x'_1 = a' + R * h' = a' + R * (b' - a')$$

$$x'_2 = b' - R * h' = b' - R * (b' - a')$$

$$= x_2 - R * (x_2 - a) = b - R * h - R * (b - R * h - a)$$

we would like to reuse one of the previous evaluations of  $E$ , so we require that  $x_1 = x'_2$ . This leads to the equation

$$R^2 - 3R + 1 = 0 \quad \text{with} \quad R = \frac{3 \pm \sqrt{5}}{2}$$

We need to choose **minus** sign since fraction  $R < 1$ .

