Fractals

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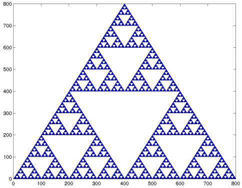
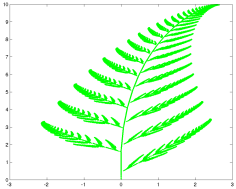
The College of William & Mary

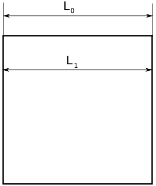
Lecture 12

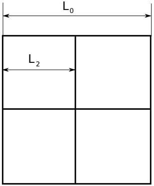


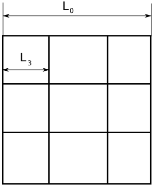


Dimension definition

Let's take a square.
What is its dimension?







$$L_1 = L_0 \qquad N_1 = 1 = 1^2$$

$$L_2 = L_0/2 \qquad N_2 = 4 = 2^2$$

$$L_3 = L_0/3 \qquad N_3 = 9 = 3^2$$

Let's define scale as $s_n = L_0/L_n$ so both N_n and s_n grow with n

Dimension

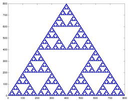
$$D = \lim_{n\to\infty} \frac{\log N_n}{\log s_n}$$

For square

$$D = \lim_{n\to\infty} \frac{\log n^2}{\log n} = 2$$

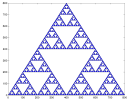
Fractional dimension object - fractal

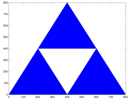
What about this figure: Sierpinski triangle
What is its dimension?



Fractional dimension object - fractal

What about this figure: Sierpinski triangle
What is its dimension?

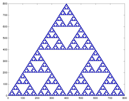


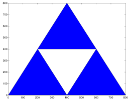


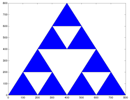
$$L_1 = L_0/2 = L_0/2^1$$
$$N_1 = 3 = 3^1$$

Fractional dimension object - fractal

What about this figure: Sierpinski triangle
What is its dimension?





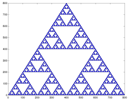


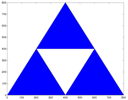
$$L_1 = L_0/2 = L_0/2^1 \qquad L_2 = L_0/4 = L_0/2^2$$

$$N_1 = 3 = 3^1 \qquad N_2 = 9 = 3^2$$

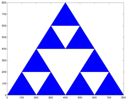
Fractional dimension object - fractal

What about this figure: Sierpinski triangle
What is its dimension?

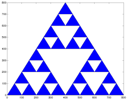




$$L_1 = L_0/2 = L_0/2^1$$
$$N_1 = 3 = 3^1$$



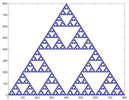
$$L_2 = L_0/4 = L_0/2^2$$
$$N_2 = 9 = 3^2$$

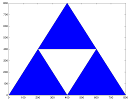


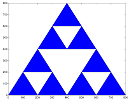
$$L_3 = L_0/8 = L_0/2^3$$
$$N_3 = 27 = 3^3$$

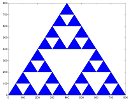
Fractional dimension object - fractal

What about this figure: Sierpinski triangle
What is its dimension?









$$L_1 = L_0/2 = L_0/2^1 \qquad L_2 = L_0/4 = L_0/2^2 \qquad L_3 = L_0/8 = L_0/2^3$$

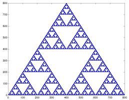
$$N_1 = 3 = 3^1 \qquad N_2 = 9 = 3^2 \qquad N_3 = 27 = 3^3$$

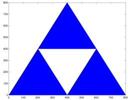
Dimension

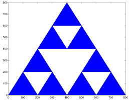
$$D = \lim_{n\to\infty} \frac{\log N_n}{\log s_n} = \lim_{n\to\infty} \frac{\log N_n}{\log L_0/L_n}$$

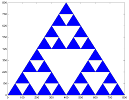
Fractional dimension object - fractal

What about this figure: Sierpinski triangle
What is its dimension?









$$L_1 = L_0/2 = L_0/2^1 \qquad L_2 = L_0/4 = L_0/2^2 \qquad L_3 = L_0/8 = L_0/2^3$$
$$N_1 = 3 = 3^1 \qquad N_2 = 9 = 3^2 \qquad N_3 = 27 = 3^3$$

Dimension

$$D = \lim_{n\to\infty} \frac{\log N_n}{\log s_n} = \lim_{n\to\infty} \frac{\log N_n}{\log L_0/L_n}$$

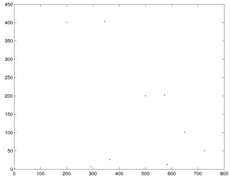
For square

$$D = \lim_{n\to\infty} \frac{\log 3^n}{\log 2^n} = \frac{\log 3}{\log 2} = 1.585$$

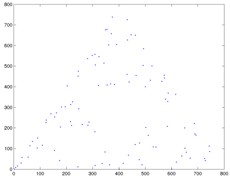
Chaos to order: fractional division - fractal

1. Choose 3 vertexes for a triangle
2. Choose random point inside the triangle
3. Choose a vertex at random
4. Mark a point half-way between known point and the chosen vertex
5. Replace coordinates of old point with this one
6. repeat from step 3

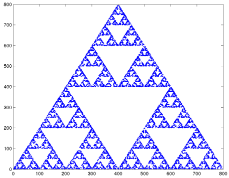
N=10



N=100



N=1000



Affine transformations

Generate a new point from the old one

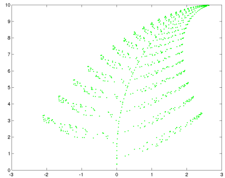
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

Old point could be

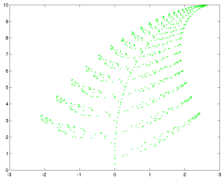
- translated
- scaled
- rotated

Example the Barnsley fern

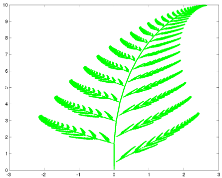
N=1000



N=10000



N=100000



Coastline length problem

Box counting algorithm
Length of the coast line

$$L_{tot} = L_n N_n$$

Recall that

$$\begin{aligned} L_n &= L_0/s_n \\ D &= = \log(N)/\log(s) \end{aligned}$$

then $N = s^D$

$$L_{tot} = \frac{L_0}{s} s^D = L_0 s^{D-1}$$

If $D > 1$ $L_{tot} = \infty$ with the scale ($s_n \sim 1/L_n$) grows with smaller and smaller box