Numerical integration continued

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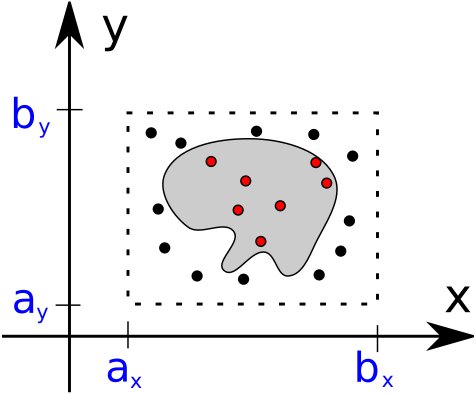
Lecture 10

Toy example - area of the pond



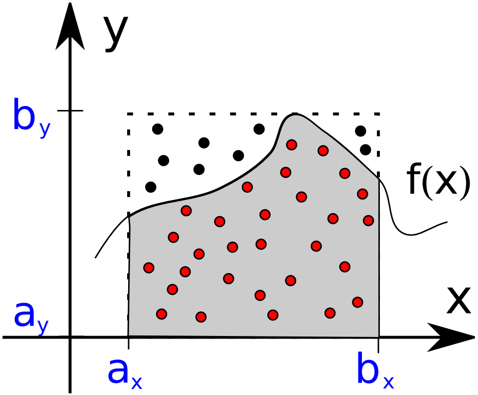
$$A_{pond} = \frac{N_{inside}}{N_{total}} A_{box}$$

where

$$A_{box} = (b_x - a_x)(b_y - a_y)$$

- Points must be uniformly and randomly distributed across the area.
- The smaller the enclosing box the better it is.

Naive Monte Carlo integration



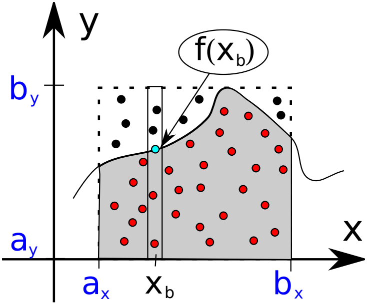
$$\int_{a_x}^{b_x} f(x)dx = \frac{N_{inside}}{N_{total}} A_{box}$$

where

$$A_{box} = (b_x - a_x)(b_y)$$

- Points must be uniformly and randomly distributed across the area.
- The smaller the enclosing box the better it is. So $max(f(x)) \to b_y$

Monte Carlo integration derived



Notice that if we choose a small stripe around the bin value x_b, then subset of points in that stripe gives an estimate for $f(x_b)$.
Thus why bother spreading points around area?

Let's chose a uniform random distribution of points x_i inside $[a_x, b_x]$

$$\int_{a_x}^{b_x} f(x)dx \approx \frac{b_x - a_x}{N} \sum_{i=1}^{N} f(x_i)$$

Error estimate for Monte-Carlo method

It can be shown that error of the numerical integration (E) is given by the following expressions

Monte Carlo method

$$E = \mathcal{O}\left((b_x - a_x) \sqrt{\frac{< f^2 > - < f >^2}{N}} \right)$$

where

$$\begin{aligned} < f > &= \frac{1}{N} \sum_{i=1}^{N} f(x_i) \\ < f^2 > &= \frac{1}{N} \sum_{i=1}^{N} f^2(x_i) \end{aligned}$$

The term under $\sqrt{}$ is the estimate of the function standard deviation.

Error estimate for other methods

Rectangle method

$$E = \mathcal{O}\left(\frac{(b_x - a_x)h}{2} f'\right) = \mathcal{O}\left(\frac{(b_x - a_x)^2}{2N} f'\right)$$

Trapezoidal method

$$E = \mathcal{O}\left(\frac{(b_x - a_x)h^2}{12} f''\right) = \mathcal{O}\left(\frac{(b_x - a_x)^3}{12N^2} f''\right)$$

Simpson method

$$E = \mathcal{O}\left(\frac{(b_x - a_x)h^4}{180} f^{(4)}\right) = \mathcal{O}\left(\frac{(b_x - a_x)^5}{180N^4} f^{(4)}\right)$$

Multidimensional integration with interval splitting

$$\int_{a_x}^{b_x} \int_{a_y}^{b_y} f(x,y)\, dx\, dy = \int_{a_x}^{b_x} dx \int_{a_y}^{b_y} dy\, f(x,y)$$

Note that last integral is the function of only x

$$\int_{a_y}^{b_y} dy\, f(x,y) = F(x)$$

$$\int_{a_x}^{b_x} \int_{a_y}^{b_y} f(x,y)\, dx\, dy = \int_{a_x}^{b_x} dx\, F(x)$$

Thus we replaced multidimensional integral as consequent series of single dimension integrals, which we already know how to do.
3D case would look like this

$$\int_{a_x}^{b_x} \int_{a_y}^{b_y} \int_{a_z}^{b_z} f(x,y,z)\, dx\, dy\, dz = \int_{a_x}^{b_x} dx \int_{a_y}^{b_y} dy \int_{a_z}^{b_z} dz\, f(x,y,z)$$

Multidimensional integration with Monte Carlo

Note that if we would like to split integration region by N points in every of D dimensions, then evaluation time grows $\sim N^D$, which renders Rectangle, Trapezoidal, Simpson, and alike method useless.
Monte Carlo method is a notable exception, it looks very simple even for multidimensional case and maintains the same $\sim N$ evaluation time.
3D case would look like this

$$\int_{a_x}^{b_x} dx \int_{a_y}^{b_y} dy \int_{a_z}^{b_z} dz\, f(x,y,z) \approx \frac{(b_x - a_x)(b_y - a_y)(b_z - a_z)}{N} \sum_{i=1}^{N} f(x_i, y_i, z_i)$$

A general case

$$\int_V dV\, f(\vec{x}) = \int_V dx_1 dx_2 dx_3 ... dx_D\, f(\vec{x}) \approx \frac{V}{N} \sum_{i=1}^{N} f(\vec{x}_i)$$

where V is the multidimensional volume , $\vec{x}_i$ randomly and uniformly distributed points in the volume V

Matlab functions for integration

1D integration

- `integral`
- `trapz`
- `quad`

2D and 3D

- `integral2`
- `integral3`

There are many others as well. See Numerical Integration help section.
Matlab's implementations are more powerful than those which we discussed but deep inside they use similar methods.