

Root finding

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Lecture 07

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Example

$$3x^3 + 2 = \sin x \quad \rightarrow \quad 3x^3 + 2 - \sin x = 0$$

Trial and error method

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A general search algorithm is the following

- make a guess i.e. trial
- make intelligent new guess (x_{i+1}) judging from this trial (x_i)
- continue as long as $|f(x_{i+1})| > \varepsilon_f$ and $|x_{i+1} - x_i| > \varepsilon_x$

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Example

Let's play a simple game:

- someone think of any number between 1 and 100
- I will make a guess
- you provide me with either “less” or “more” depending where is my guess with respect to your number

How many guesses do I need?

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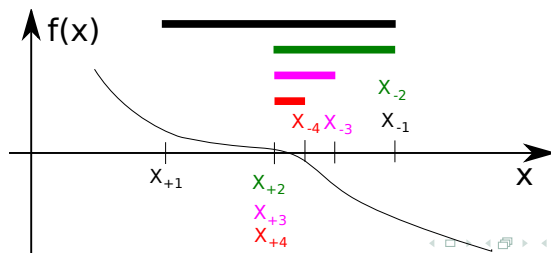
- someone think of any number between 1 and 100
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How many guesses do I need? At most **7**

Bisection method pseudo code

Works for any **continuous function** in vicinity of a function root

- make initial bracket for search, i.e. set x_+ and x_- such that
 - $f(x_+) > 0$
 - $f(x_-) < 0$
- loop begins
- make the new guess value $x_g = (x_+ + x_-)/2$
- if $|f(x_g)| \leq \varepsilon_f$ and $|x_+ - x_g| \leq \varepsilon_x$
then stop, we found the solution with the desired precision
- otherwise if $f(x_g) > 0$ then $x_+ = x_g$ else $x_- = x_g$
- continue the loop



Bisection - simplified matlab implementation

```
function x_sol=bisection(f, xn, xp, eps_f, eps_x)
% solving f(x)=0 with bisection method

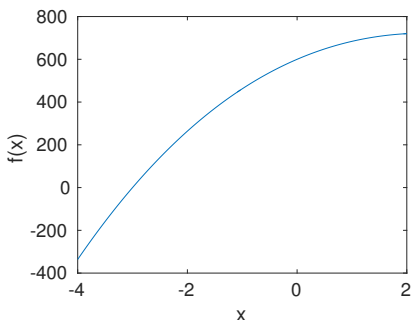
xg=(xp+xn)/2; % initial guess
fg=f(xg);     % initial function evaluation

while ( (abs(fg) > eps_f) || (abs(xg-xp)>eps_x) )
    if (fg>0)
        xp=xg;
    else
        xn=xg;
    end
    xg=(xp+xn)/2; % update guess
    fg=f(xg);     % update function evaluation
end
x_sol=xg; % solution is ready
end
```

Bisection - example of use

Let's define a simple test function in the file `function_to_solve.m`

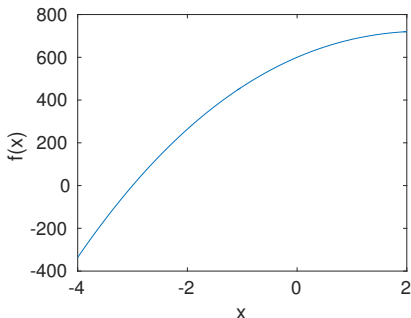
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function ret=function_to_solve(x)
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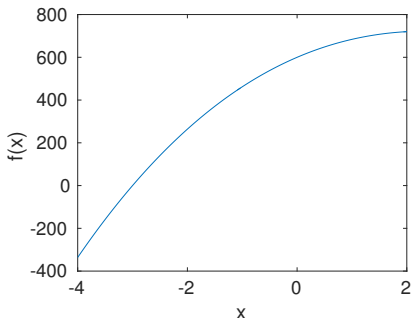
pay attention to the function handle operator **@**

```
eps_x=1e-8;
eps_f=1e-6;
x0=bisection(...
    @function_to_solve,...
    -4.1, 2, ...
    eps_f, eps_x)
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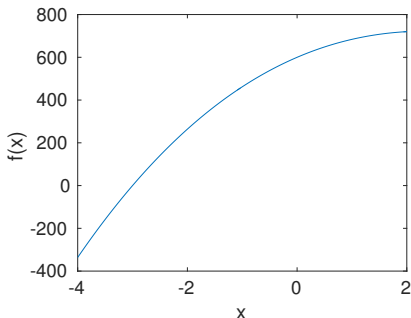
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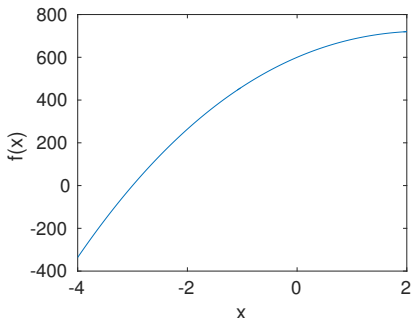
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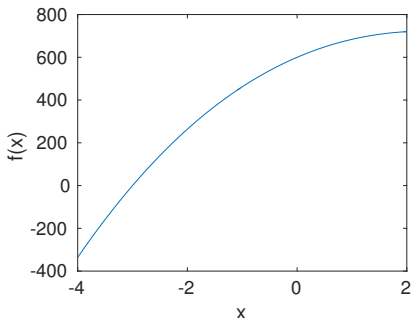
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So what should we check for sure

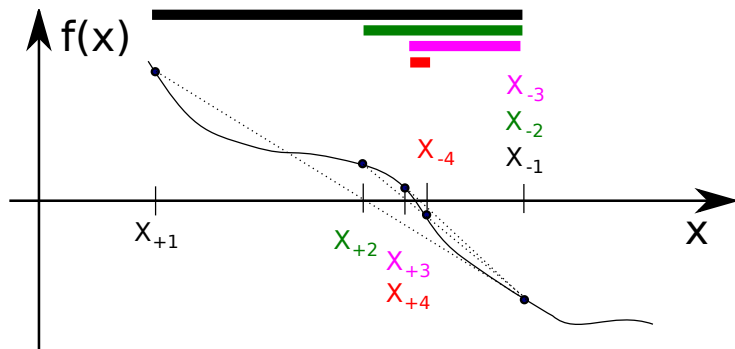
- 1 $f(xn) < 0$
- 2 $f(xp) > 0$

It would be handy to return secondary outputs

- the value of the function at the found solution point
- the number of iterations (good for performance tests)

False position (*regula falsi*) method

In this method we naively approximate our function as a line.



False position method - pseudo code

- make initial bracket for search x_+ and x_- such that
 - $f(x_+) > 0$
 - $f(x_-) < 0$
- loop begins
- draw a chord between points $(x_-, f(x_-))$ and $(x_+, f(x_+))$
- make new guess value at the point of the chord intersection with the 'x' axis

$$x_g = \frac{x_- f(x_+) - x_+ f(x_-)}{f(x_+) - f(x_-)}$$

- if $|f(x_g)| \leq \varepsilon_f$ and ($|x_+ - x_g| \leq \varepsilon_x$ or $|x_- - x_g| \leq \varepsilon_x$)
stop we found the solution with desired approximation
- otherwise if $f(x_g) > 0$ then $x_+ = x_g$ else $x_- = x_g$
- continue the loop

Note: the only difference from the bisection is the way of updating x_g and the way to check the x_g convergence

Solution convergence

We say that algorithm has defined convergence if it is possible to express

$$\lim_{k \rightarrow \infty} (x_{k+1} - x_0) = c(x_k - x_0)^m$$

Where x_0 is true root of the equation, c is some constant, and m is the order of convergence.

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- In general, it is impossible to define the convergence order for the false position method

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Generally the speed of the algorithm is related to its convergence order. However, other factors may affect the speed.