# Root finding

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Lecture 07



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**Example**

$$3x^3 + 2 = \sin x \quad \to \quad 3x^3 + 2 - \sin x = 0$$



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However, each guess can provide some clues.

**A general search algorithm is the following**

- make a guess i.e. trial
- make intelligent new guess ($x_{i+1}$) judging from this trial ($x_i$)
- continue as long as $|f(x_{i+1})| > \varepsilon_f$ and $|x_{i+1} - x_i| > \varepsilon_x$



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**Example**

Let's play a simple game:

- someone think of any number between 1 and 100
- I will make a guess
- you provide me with either "less" or "more" depending where is my guess with respect to your number

How many guesses do I need?



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How many guesses do I need? At most 7



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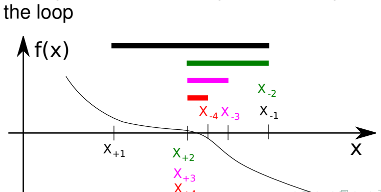
## Bisection method pseudo code

Works for any continuous function in vicinity of a function root

- make initial bracket for search, i.e. set $x_+$ and $x_-$ such that
  - $f(x_+) > 0$
  - $f(x_-) < 0$
- loop begins
- make the new guess value $x_g = (x_+ + x_-)/2$
- if $|f(x_g)| \le \varepsilon_f$ and $|x_+ - x_g| \le \varepsilon_x$
  then stop, we found the solution with the desired precision
- otherwise if $f(x_g) > 0$ then $x_+ = x_g$ else $x_- = x_g$
- continue the loop






Notes

## Bisection - simplified matlab implementation

```
function x_sol=bisection(f, xn, xp, eps_f, eps_x)
% solving f(x)=0 with bisection method

  xg=(xp+xn)/2; % initial guess
  fg=f(xg);     % initial function evaluation

  while ( (abs(fg) > eps_f) || (abs(xg-xp)>eps_x) )
    if (fg>0)
      xp=xg;
    else
      xn=xg;
    end
    xg=(xp+xn)/2; % update guess
    fg=f(xg);     % update function evaluation
  end
  x_sol=xg; % solution is ready
end
```



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## Bisection - example of use

Let's define a simple test function in the file `function_to_solve.m`

```
function ret=function_to_solve(x)
  ret=(x-10).*(x-20).*(x+3);
end
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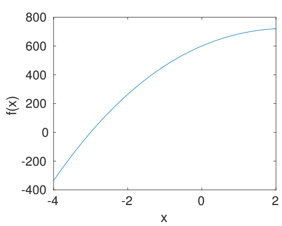




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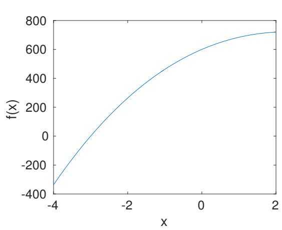


pay attention to the function handle operator @

```
eps_x=1e-8;
eps_f=1e-6;
x0=bisection(...
  @function_to_solve,...
  -4.1, 2, ...
  eps_f, eps_x)
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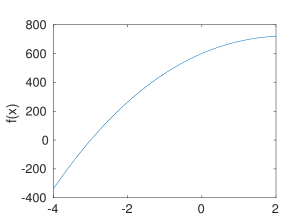


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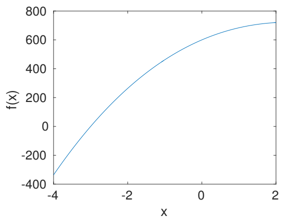


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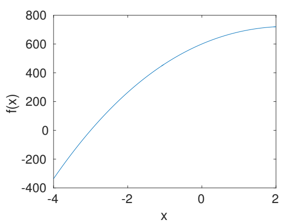

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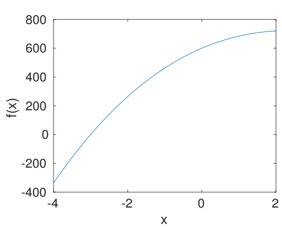

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So what should we check for sure

1. $f(xn) < 0$
2. $f(xp) > 0$

It would be handy to return secondary outputs

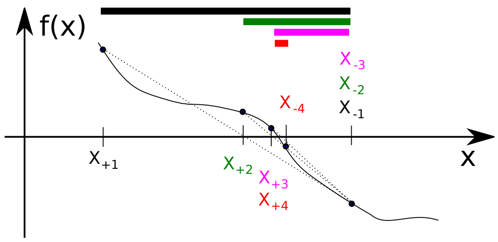
- the value of the function at the found solution point
- the number of iterations (good for performance tests)



Notes

## False position (*regula falsi*) method

In this method we naively approximate our function as a line.



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## False position method - pseudo code

- make initial bracket for search $x_+$ and $x_-$ such that
  - $f(x_+) > 0$
  - $f(x_-) < 0$
- loop begins
- draw a chord between points $(x_-, f(x_-))$ and $(x_+, f(x_+))$
- make new guess value at the point of the chord intersection with the 'x' axis

$$x_g = \frac{x_- f(x_+) - x_+ f(x_-)}{f(x_+) - f(x_-)}$$

- if $|f(x_g)| \le \varepsilon_f$ and ($|x_+ - x_g| \le \varepsilon_x$ or $|x_- - x_g| \le \varepsilon_x$) stop we found the solution with desired approximation
- otherwise if $f(x_g) > 0$ then $x_+ = x_g$ else $x_- = x_g$
- continue the loop

Note: the only difference from the bisection is the way of updating $x_g$ and the way to check the $x_g$ convergence



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## Solution convergence

We say that algorithm has defined convergence if it is possible to express

$$\lim_{k \to \infty} (x_{k+1} - x_0) = c(x_k - x_0)^m$$

Where $x_0$ is true root of the equation, $c$ is some constant, and $m$ is the order of convergence.



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Generally the speed of the algorithm is related to its convergence order. However, other factors may affect the speed.



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