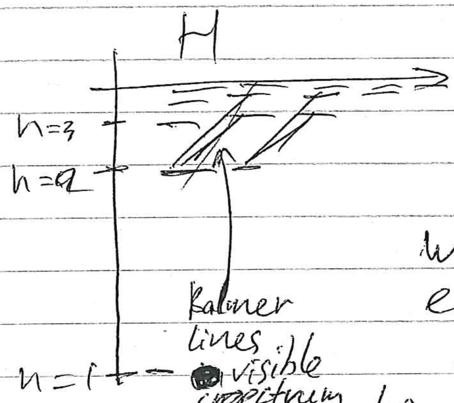
lecture 13 Disappearing spectral lines and Saha equation

it was noticed that many spectral lines appear and then dissappear as one sort the stars spectra in increasing temperature order.

Luck of line at small temperature corresponding to transitions from non ground level is not a big puzle:



For example for Balmer lines corresponding to absorption of a photon from $n=2$ to $n' > 2$

We need to have some electron population (i.e. probab. of state to be occupied) to be quite significant

Boltzmann distribution dictates

$$\frac{P(n=2)}{P(n=1)} = \frac{\frac{1}{Z}\, g_{n=2}\, e^{-E_2/kT}}{\frac{1}{Z}\, g_{n=1}\, e^{-E_1/kT}} = \frac{2\cdot 4}{2\cdot 1} e^{-\frac{(E_2-E_1)}{kT}}$$

degeneracy of a state with $n=2,1$

$$g_n = \underbrace{2}_{spin} \cdot \underbrace{n^2}_{degeneracy}$$

$$Z = \sum_{n=1}^{\infty} g_n e^{-E_n/kT}$$

(p2)

$$\frac{P(n=2)}{p(n=1)} = 4 \cdot e^{-\left(-\frac{13.6eV}{4} + \frac{13.6eV}{1}\right)\frac{1}{KT}}$$

$$= 4 \cdot e^{-\frac{3}{4}\frac{13.6eV}{KT}} =$$

/ 1eV = 11'600 K /

$$= 4 \cdot e^{-\frac{3}{4}\cdot\frac{13.6\cdot 11600}{T}} = 4e^{-118000/T}$$

In order to get any reasonable population at $n=2$ we need huge Temperatures $\boxed{T\sim 10^5}$, while stars ~~even~~ the hot one have $\boxed{T \lesssim 40000K}$.

Apperance of the line from $n=3 \to n>3$ is even less probable

Note: it seems that for high n

$$\frac{p(n\to\infty)}{p(n=1)} = \frac{2\cdot n^2 e^{-E_\infty/KT}}{2\cdot e^{-E_1/KT}} \sim \underset{\text{goes to }\infty}{n^2}\ \underset{\text{fixed}}{\boxed{e^{-\frac{13.6eV}{KT}}}}$$

so we might think that ~~we~~ high 'n' levels might be well populated compare to the ground levels but

$n=\infty$ is unrealistic
size of the electron orbit grows as $\boxed{r_n \sim a_0 n^2}$ for Hydrogen
~~a~~ so high 'n' is ~~ap~~ ~~a~~ unphysical

So far so good, so we see that line should increase it strength (~~also~~ absorption grows) with higher temperature but why it disappear at high T ?

Idea 1, hot stars are made of not hydrogen. But what would be ~~the mechanism~~ ?

Idea 2: neutral Hydrogen replaced with its ion.

$H \rightarrow H^+ + e^-$, no electron around proton then no ~~problem~~ absorption ~~since~~ at the former neutral hydrogen transition.

if we call N_{i+1} - number of atoms with i electron removed
i.e. $i \Leftrightarrow$ ionization level

partition function

Saha eq.

$$\boxed{\frac{N_{i+1}}{N_i} = \frac{2 Z_{i+1}}{n_e Z_i}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_i/T}}$$

electron density

χ_i - ionization energy $i \rightarrow i+1$

$$Z_i = \sum_n g_n e^{\frac{-E_{n_i} - E_{g_i}}{kT}}$$

m_e - electron mass
/ n_e - electron density

ideal gas $p_e = n_e kT$
pressure

Saha equation

number of 'particles' in state i
vs number of particles in energy state j

is given by

$$\frac{N_j}{N_i} = \frac{g_j e^{-E_j/kT}}{g_i e^{-E_i/kT}}$$, where g is degeneracy

Now let's compare an atom in initial non ionized state 'i' and when one ion removed '$i+1$'

in this case we need to add up probabilities to find atom in its all possible states

$$Z_i = \sum_{m=i}^{\infty} g_m e^{-E_m/kT}$$ where we count E_m from the ground level

But for ion it is a bit more complex

$$Z_{\text{ionization}} = Z_{\text{atom ionized}} \cdot Z_{\text{electron}}$$

$Z_{\text{electron}} \sim g_v \int e^{-\frac{m_e v^2}{2}/kT} dv = 2 \frac{1}{n_e} \left(\frac{2\pi m_e kT}{h^2} \right)^{3/2}$

2 — spin $\pm\frac{1}{2}$

$= Z_{i+1} e^{-\chi_i/kT}$

Z_{i+1} — energy counted from ion ground level

χ_i — ionization energy '$i+1$' from 'i' ground level ratio of next level ionized atoms to previous ionization

$$\boxed{\frac{N_{i+1}}{N_i} = \frac{2}{n_e}\cdot\frac{Z_{i+1}}{Z_i}\, e^{-\chi_i/kT}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2}}$$

(atom / ion) levels only

for ideal gas $\boxed{P_e = n_e kT}$

Now we see that high temperatures favor <u>ionization</u>.

So lets estimate ionization for hydrogen only star

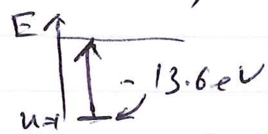
H_I - neutral , H_{II} - first level ion

$$Z_i = Z_{H_I} = \sum_{n=1}^{\infty} \underset{2n^2}{g_n}\, e^{-\left(\frac{E_n - E_g}{kT}\right)} \approx$$

$$\approx 2e^{-0/kT} + \underbrace{\sum_{n=2}^{\infty} 2n^2 e^{-\frac{E_n - E_g}{kT}}}_{\to 0}$$

$Z_{i+1} = 1$ since ionized Hydrogen is just a proton with only one state

$\chi_i = -13.6\,eV$

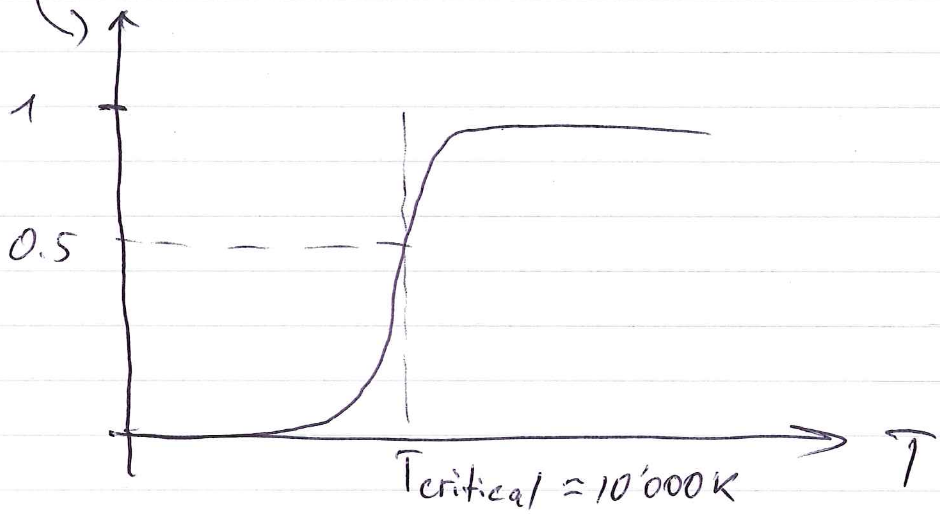


P_e - hard to get, but from measurement it is 0.1 N/m2

$$\frac{N_{H_{II}}}{N_{H_I}} =$$

$$\frac{N_{H\,II}}{N_{H\,I}} = \frac{2}{P_e}\, kT \, \frac{1}{2} \cdot e^{-\frac{\chi_i}{kT}} \cdot \left(\frac{2\pi m_e}{h^2} kT \right)^{3/2}$$

$$\frac{N_{II}}{N_{Total}} = \frac{N_{II}}{N_I + N_{II}} = \frac{N_{H\,II}/N_{H\,I}}{1 + \frac{N_{H\,II}}{N_{H\,I}}}$$



$N_2 \leftarrow$ first exited level of Hydrogen

$$\frac{N_2}{N_{Total}} = \frac{N_2}{N_1 + N_2} \; \frac{\overbrace{N_1 + N_2}^{\approx N_{H\,I}}}{\underbrace{N_{Total}}_{N_{H\,I} + N_{H\,II}}} =$$

Recall
$B_{\alpha} = 656\,nm$
↑
red

$$= \frac{N_2/N_1}{1 + N_2/N_1} \cdot \frac{1}{1 + \frac{N_{H\,II}}{N_{H\,I}}}$$

$$'' \quad \frac{e^{-\frac{10eV}{kT}}}{1 + e^{-\frac{10eV}{kT}}} \cdot \frac{1}{1 + \frac{N_{H\,II}}{N_{H\,I}}}$$