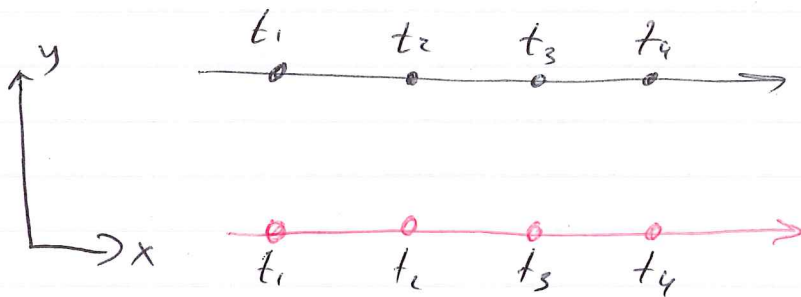


Lecture 07

(P1)

Some extra trivia:

Q1: For this trajectories, are bodies gravitationally bounded

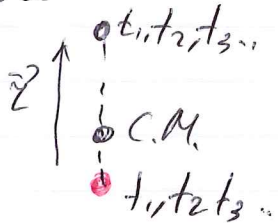


A₁: ^{naive} no! Because C.M. seems to move

A₂: But we are not in C.M. ref. frame!

C.M. must be on a line connecting
2 bodies

Possible C.M. ref frame, recall $\vec{z}_{cm} = \frac{m_1 \vec{x}_1 + m_2 \vec{x}_2}{m_1 + m_2}$

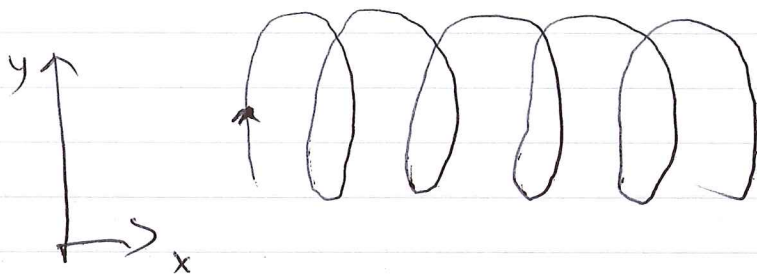


So there is no ^{apparent} orbit
around C.M.

So either $|z| = \infty$
or they are not
gravitationally bounded

Q2:

what if we can track one body which does this.



Q2a: Is it even possible?

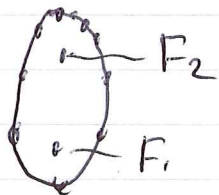
A: Yes, if C.M. moves along x with constant velocity which is totally fine

So in C.M. ref. frame it will look like



Q2b: can we say where is C.M.?

A2b: No! There are 2 possible locations:



Without observing 2nd body we have no clue.

Or we need to time 1st body orbit. It moves faster near "perihelion" i.e. focus.

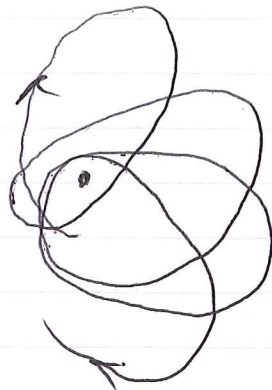
Q3: What if more than 2 ~~gravitationally~~
~~bounded~~ bodies,

A: The problem has no analytical
solution $\tilde{}$

Bring your computer and solve
it numerically (some might
remember my midterm in Phys 256)

If 3rd mass relatively light,
we can treat it as perturbation

so original orbit (2 body case)
will precess



This is mostly
the case for sun - planets orbits
in our solar system.

Halley's comet problem (2.14 from the book)

T = 76 years ← orbital period

e = 0.9673

Find 'a'

Well 3rd Kepler's Law

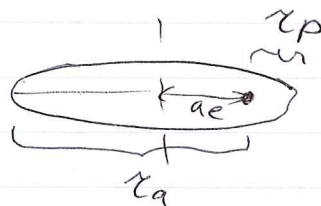
T^2 = a^3
↑ in years ↑ in au

Note so far experimental observ. only

⇒ ~~a~~ note that by Kepler's law this is average distance

a ≈ ∛(76^2) = 18 au

b ≈ a √(1-e^2) = 4.55 au



aphelion distance from sun

ra = a + ae = a(1+e) ≈ 35.3 AU

perihelion

rp = a - ae = a(1-e) ≈ 0.59 AU

(P5)

Ok, what about mass of the system?

Naively: let's use 'e' expression ...

But 'e' is a bad parameter:
many orbits have the same 'e'.

Think about circular orbits ($e=0$)
around heavy body.

So to find masses, we need
to go ~~deeper~~ deeper into

3rd Kepler's Law derivation,

(p0)

Recap

$$E = \frac{1}{2}(\dot{r})^2 + \frac{L^2}{2\mu r^2} - \frac{\alpha}{r}$$

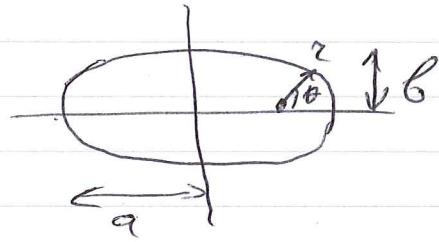
$$L = \mu r^2 \dot{\theta} = 2\mu (\dot{A})$$

$$\frac{r}{z} = 1 + e \cos \theta$$

$$a = \frac{p}{1-e^2}; b = \frac{p}{\sqrt{1-e^2}}$$

$$b^2 = a^2(1-e^2)$$

~~etc~~



3rd Kepler's Law

(Or how ~~was~~ do we know the mass of the sun?)

Recall that

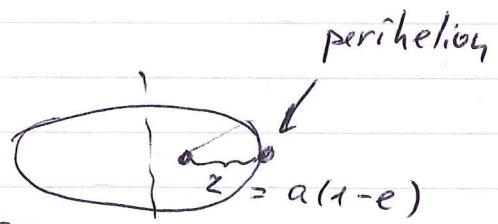
$$L = \mu r^2 \dot{\theta} = \mu 2(dA)$$

$$\Rightarrow \int dA = \int \frac{L}{2\mu} dt$$

↙ period

$$\pi a b = \frac{L}{2\mu} \cdot T$$

$$\pi \frac{P}{1-e^2} \cdot \frac{P}{\sqrt{1-e^2}} = \frac{L}{2\mu} T$$



$$\pi a b = \pi a^2 \sqrt{1-e^2} = \pi a^2 \frac{2EL}{\mu r^2} = \frac{L}{2\mu} T$$

looking for L and μ from energy conservation p.v.

$$E = \frac{\mu (\dot{r})^2}{2} + \frac{L^2}{2\mu r^2} - \frac{\alpha}{r}$$

$(\dot{r}) = 0$ in perihelion

$$\Rightarrow E = \frac{L^2}{2\mu r^2} - \frac{\alpha}{r} \quad (r = a(1-e))$$

$$\Rightarrow E = \begin{cases} \frac{L^2}{2\mu a^2(1-e)^2} - \frac{\alpha}{a(1-e)} \\ \frac{L^2}{2\mu a^2(1+e)^2} - \frac{\alpha}{a(1+e)} \end{cases} \quad \swarrow \text{aphelion}$$

$$T = \frac{2\pi a^2 \cdot 2EL}{\alpha^2}$$

$$\frac{L^2}{2\mu a^2} \frac{1}{(1-e)^2} - \frac{\alpha}{a(1-e)} = \frac{L^2}{2\mu a^2} \frac{1}{(1+e)^2} - \frac{\alpha}{(1+e)a}$$

$$\frac{L^2}{2\mu a} \frac{1}{1-e} \left(\frac{1}{1-e} - \alpha \right) = \frac{L^2}{2\mu a} \frac{1}{1+e} \left(\frac{1}{1+e} - \alpha \right)$$

~~$$\frac{L^2}{2\mu a} \frac{e}{(1-e)^2} = \frac{L^2}{2\mu a} \frac{(-e)}{(1+e)^2}$$~~

$$\frac{L^2}{2\mu a} \left[\frac{1}{(1-e)^2} - \frac{1}{(1+e)^2} \right] = \alpha \left[\frac{1}{1-e} - \frac{1}{1+e} \right]$$

$$\frac{L^2}{2\mu a} \frac{4e}{(1-e^2)^2} = \frac{\alpha 2e}{(1-e^2)}$$

$$L = \sqrt{\mu a \alpha (1-e^2)}$$

recall now

$$\pi a^2 \sqrt{1-e^2} = \frac{L}{2\mu} T = \frac{\sqrt{\mu a \alpha (1-e^2)}}{2\mu} T$$

$$\frac{2\pi a^2 \sqrt{\mu}}{\sqrt{a\alpha}} = T$$

$$4\pi^2 a^3 \left(\frac{\mu}{\alpha} \right) = 4\pi^2 a^3 \frac{m_1 m_2}{G(m_1+m_2)} =$$

$$\Rightarrow \boxed{\frac{4\pi^2 a^3}{G(m_1+m_2)} = T^2} \quad \text{3rd Kepler's law}$$

So 3rd Kepler's Law is very powerful tool to find total mass of 2 body system.

This how we know mass of Sun.

$$T = 1 \text{ year} \approx \pi \cdot 10^7 \text{ sec}$$

$$a \approx 1 \text{ au} \approx 1.5 \cdot 10^{11} \text{ m} \leftarrow \text{can be done with parallax observations}$$

$$G = 6.7 \cdot 10^{-11} \text{ Nm}^2/\text{kg}^2$$

$$T^2 = \frac{4\pi^2 a^3}{G(m_{\odot} + m_{\oplus})}$$

$$M_T = \frac{4\pi^2 a^3}{G \cdot T^2} = \frac{4 \cdot (\pi)^2 \cdot (1.5 \cdot 10^{11})^3}{6.7 \cdot 10^{-11} \cdot (\pi \cdot 10^7)^2}$$

$$\approx 20 \cdot 10^{33+11} = 2 \cdot 10^{45}$$

$$\approx \frac{4 \cdot 3.6 \cdot 10^{33}}{6.7 \cdot 10^{-11}} = 2 \cdot 10^{30} \text{ kg} = M_T$$

$$2 \cdot 10^{30} \text{ kg} = M_T$$

Even for Jupiter it is hard. We can use moons but of Jupiter to find its mass.

But what is the mass of ~~the~~ Earth?

We cannot do it from $\odot \leftrightarrow \oplus$ system alone. Not enough precision in known numbers even if we can track sun and Earth orbit to figure out a_{\odot} and a_{\oplus}

and relate it to Mass ratio.

Q: So what to do? A: use Earth-Moon system

3rd Kepler's law revisited.
Interesting numerology ☺

$$T^2 = \frac{4\pi^2 a^3}{G(M_T)}$$

T : s → year
 a : m → AU

$$T = T_y \cdot \frac{\text{sec in year}}{\text{year}}$$

$$a = a_{au} \cdot \frac{\text{m per au}}{1 \text{ au}}$$

$$\left[T_y \cdot \left(\pi \cdot 10^7 \frac{\text{s}}{\text{y}} \right) \right]^2 = \frac{4\pi^2 \left(a_{au} \cdot 1.5 \cdot 10^{11} \frac{\text{m}}{\text{au}} \right)^3}{G M_T}$$



$$\frac{T_y^2}{a_{au}^3} = \frac{4\pi^2 (1.5 \cdot 10^{11})^3}{\pi^2 \cdot (10^7)^2 \cdot G \cdot M_T}$$

↑
2 · M_⊙

$$= \frac{4 \cdot \pi^2 (1.5 \cdot 10^{11})^3}{(\pi \cdot 10^7)^2 \cdot 6.7 \cdot 10^{-11} \cdot 2 \cdot 10^{30}}$$

// is it a coincidence? No.

$$\frac{T_y^2}{a_{au}^3} \approx 1/d \quad , \quad d = \frac{M_T}{M_{\odot}}$$

Q: if period of Jupiter is 11.86 years
 what is ~~distance~~ its semi major axis?

$$a \approx \sqrt[3]{(12)^2} = \sqrt[3]{144} \approx 5.2 \text{ au}$$

Q: If sun were 25 times heavier,
 what would be Earth orbit period
 A: ⇒ 1/5 of a year

↑
 which is the
 book value