# Discrete Fourier Transform and filters

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Lecture 24

# DFT vs. Matlab FFT

DFT

$$
\begin{aligned}
y_k &= \frac{1}{N}\sum_{n=0}^{N-1} c_n \exp(i\frac{2\pi(k-1)n}{N}) \quad \text{inverse Fourier transform} \\
c_n &= \sum_{k=1}^{N} y_k \exp(-i\frac{2\pi(k-1)n}{N}) \quad \text{Fourier transform} \\
n &= 0, 1, 2, \cdots, N-1
\end{aligned}
$$

Matlab FFT

$$
\begin{aligned}
y_k &= \frac{1}{N}\sum_{n=1}^{N} c_n \exp(i\frac{2\pi(k-1)(n-1)}{N}) \quad \text{inverse Fourier transform} \\
c_n &= \sum_{k=1}^{N} y_k \exp(-i\frac{2\pi(k-1)(n-1)}{N}) \quad \text{Fourier transform} \\
n &= 1, 2, \cdots, N
\end{aligned}
$$

So do DFT $\rightarrow$ Matlab FFT is equivalent of $n \rightarrow n+1$ and vice versa

# Warning about notation

$c_0$ has a special meaning. It is the 0 frequency (i.e., DC) amplitude of the signal. Thus, I will always use the DFT notation unless mentioned otherwise.

People often denote the forward Fourier transform as $\mathcal{F}$

$$Y = \mathcal{F}y$$

So $Y = (Y0, Y1, Y2, \ldots, Y_{N-1}) = (c_0, c_1, c_2, \ldots, c_{N-1})$ is the spectrum of the time domain signal $y$

Inverse Fourier transform is denoted as $\mathcal{F}^{-1}$

$$y = \mathcal{F}^{-1}Y$$

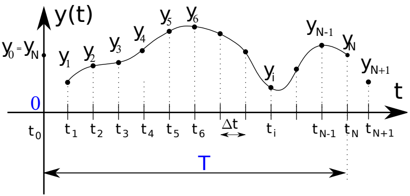
Instead of using $c_n$ coefficients, we refer in this notation to $Y_n$

# Sampling rate and important physics relationship

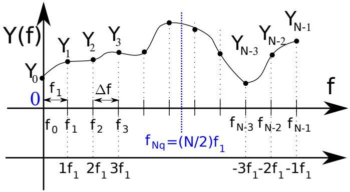
For DFT we need to have equidistant points and the signal repeating itself. We consider signals which start at time 0 and take N points over the period time $T$, thus, $y_k = y_{k+N}$. To deduce the time of a data point, we just multiply its index by the time spacing $\Delta t = T/N$. I.e., $y_i$ is taken at time $t_i = i\Delta t = i/f_s$

The sampling rate $f_s$ is defined as $f_s = 1/\Delta t = f_1 N$, and $f_1 = T/N$ is the frequency spacing in the spectrum,sometimes it is referred as the resolution bandwidth (RBW).

Time series




Spectrum




In Matlab `fft`, $Y_n$ has the frequency $f_n = f_1 \times (n-1) = f_s \times (n-1)/N$.

# Nyquist frequency

If we take $N$ data points with the sampling rate $f_s$, what is the maximum frequency which we can expect to see in our spectrum?
Naively, we can say $(N-1) \times f_1 \approx f_s$, since in the DFT spectrum all points are separated by the fundamental frequency $f_1 = 1/T = f_s/N$
However, recall that

$$Y_n = c_n = \sum_{k=1}^{N} y_k \exp(-i\frac{2\pi(k-1)n}{N})$$

Thus, $Y_{N-n} = Y_{-n}$, i.e., the higher half of the vector $Y$ contains negative frequency. So at most, we can hope to obtain a spectrum with the highest frequency smaller than

## Nyquist frequency

$$F_{Nq} = f_1\frac{N}{2} = \frac{f_s}{2}$$

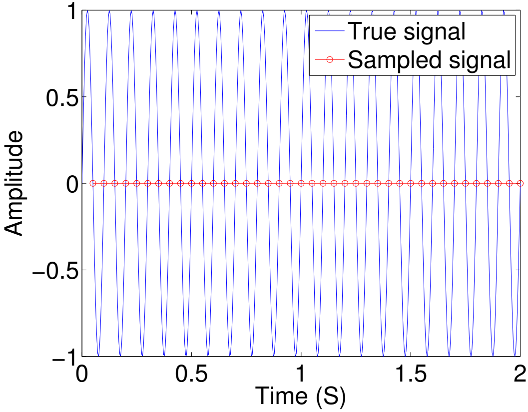
# Nyquist criteria

$$f_s > 2f_{signal}$$

You must sample your signal twice faster than the highest frequency component of it. I.e., the Nyquist frequency of your sample should be $>$ than the highest signal frequency.

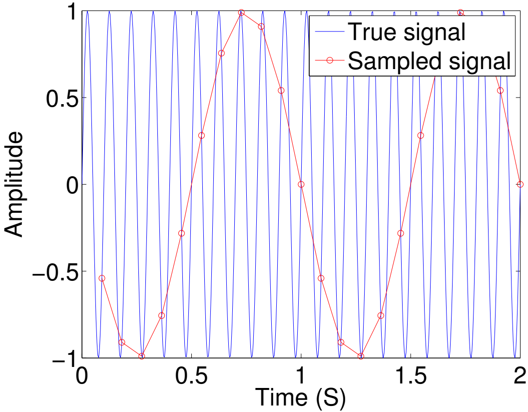
# Aliasing: wrong/slow sampling frequency

Sampling with
$f_s = 2 f_{signal}$
i.e.
$f_{Nq} = f_{signal}$
Sampled signal appeared to be DC



# Aliasing: too slow sampling frequency - reflection

Under sampling
$f_s = 1.1 f_{signal}$
The sampled signal seems to have a lower frequency.

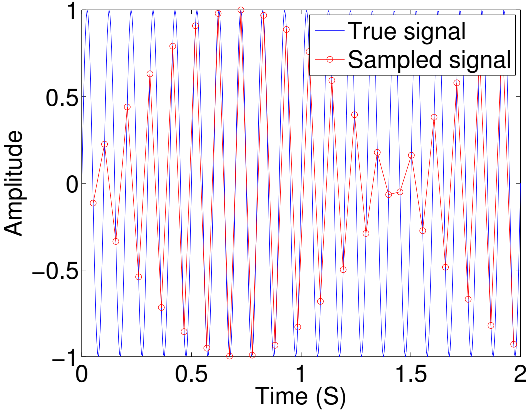



The sampled signal appears to have a slower frequency. This is case of the reflection/folding where the signal frequency is slightly higher than the sampling frequency.

$f_{apparent\ signal} = (f_{signal} - 2f_{Nq}) \approx f_{signal} - f_s$

# Aliasing: too slow sampling frequency - ghosts

Under sampling
$f_s = 1.93 f_{signal}$
The sampled signal looks very different.



# DFT filters

Once you get a signal, you can filter the unwanted frequencies out of it. The recipe is the following

- sample the signal
- calculate DFT (use Matlab `fft`)
- have a look at the spectrum and decide which frequencies are unwanted
- apply a filter which attenuate unwanted frequencies amplitudes
  - If you attenuate the component of the frequency $f$ by $g_f$, you need to attenuate the component at $-f$ by $g_f^*$. Otherwise, the inverse Fourier transform will have non zero imaginary part.
- calculate inverse DFT (`ifft`) of the filtered spectrum
- repeat if needed

Applications

- Noise reduction
- Compression