Fourier transform

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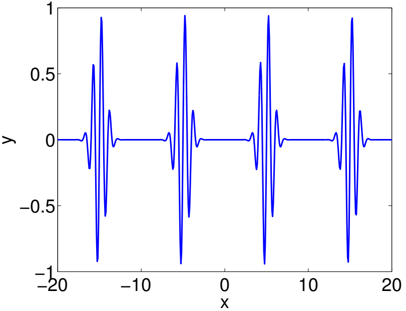
Lecture 23

Any periodic single value function with a finite number of discontinuities, and for which $\int_0^T |f(t)|dt$ is finite, can be presented as



$$y(t) = \frac{a_0}{2} + \sum_1^{\infty} (a_n \cos(n\omega_1 t) + b_n \sin(n\omega_1 t))$$

T is the period, i.e., $y(t) = y(t + T)$

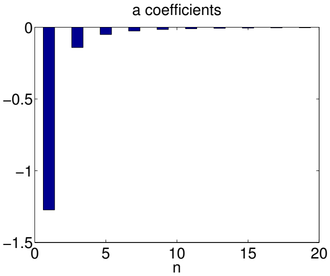
$\omega_1 = 2\pi/T$ is the fundamental frequency

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \frac{2}{T} \int_0^T dt \begin{pmatrix} \cos(n\omega_1 t) \\ \sin(n\omega_1 t) \end{pmatrix} y(t)$$

At a discontinuity, the series approaches the mid point

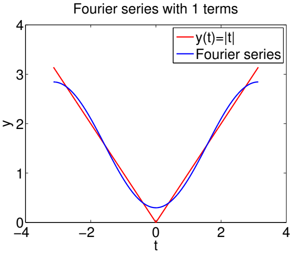
Fourier series example: |t|

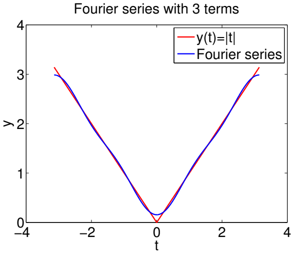
$$y(t) = |t|, \quad -pi < t < pi$$

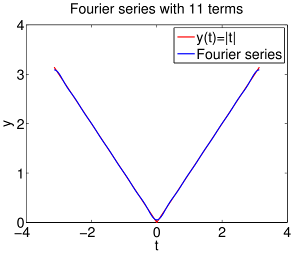


Since the function is even all $b_n = 0$

$$\begin{cases} a_0 = \pi, & \\ a_n = 0, & n \text{ is even} \\ a_n = -\frac{4}{\pi n^2}, & n \text{ is odd} \end{cases}$$





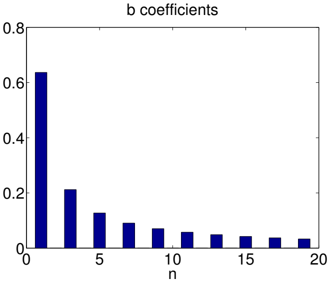


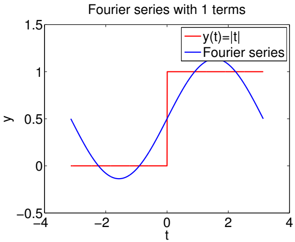
Fourier series example: step function

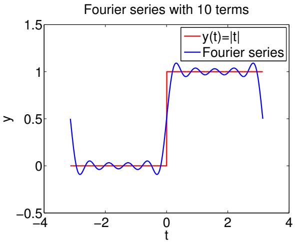
$$\begin{cases} 0, & -\pi < x < 0, \\ 1, & 0 < x < \pi \end{cases}$$

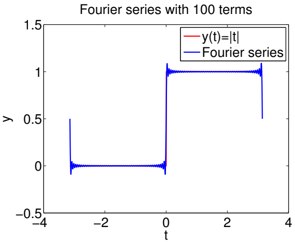
Since the function is odd all $a_n = 0$ except $a_0 = 1$

$$\begin{cases} b_n = 0, & n \text{ is even} \\ b_n = \frac{2}{\pi n}, & n \text{ is odd} \end{cases}$$









Complex representation

Recall that

$$\exp(i\omega t) = \cos(\omega t) + i\sin(\omega t)$$

It can be shown that

$$y(t) = \sum_{n=-\infty}^{\infty} c_n \exp(in\omega_1 t)$$

$$c_n = \frac{1}{T}\int_0^T y(t)\exp(-i\omega_1 nt)dt$$

$$a_n = c_n + c_{-n}$$

$$b_n = i(c_n - c_{-n})$$

What to do if function is not periodic?

- $T \to \infty$
- $\sum \to \int$
- discrete spectrum $\to$ continuous spectrum
  - $c_n \to c_\omega$

$$
\begin{aligned}
y(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c_\omega \exp(i\omega t) d\omega \\
c_\omega &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y(t) \exp(-i\omega t) dt
\end{aligned}
$$

Above requires that $\int_{-\infty}^{\infty} dt\ y(t)$ exists and is finite.

Note that c_ω has the extra $\sqrt{2\pi}$ when compared to c_n, and T is gone.

Discrete Fourier transform (DFT)

In reality, we cannot have

- infinitely large interval
- infinite amount of points to calculate true integral

Assuming that $y(t)$ has a period T and we took N equidistant points

$$\begin{aligned}
\Delta t &= \frac{T}{N} \text{ samples spacing, } f_s = \frac{1}{\Delta t} \text{ sampling rate} \\
f_1 &= \frac{1}{T} = \frac{1}{N\Delta t} \text{ smallest observed frequency,} \\
& \qquad \text{also resolution bandwidth} \\
t_k &= \Delta t \times (k-1) \\
y(t_{k+N}) &= y(t_k) \text{ periodicity condition} \\
y_k &= y(t_k) \text{ shortcut notation} \\
y_1, y_2, y_3, \cdots, y_N & \qquad \text{data set}
\end{aligned}$$

We replace the integral in the Fourier series with the sum

DFT

$$
\begin{aligned}
y_k &= \frac{1}{N}\sum_{n=0}^{N-1} c_n \exp(i\frac{2\pi(k-1)n}{N}) \quad \text{inverse Fourier transform} \\
c_n &= \sum_{k=1}^{N} y_k \exp(-i\frac{2\pi(k-1)n}{N}) \quad \text{Fourier transform} \\
n &= 0, 1, 2, \cdots, N-1
\end{aligned}
$$

Confusion keeps increasing: where are the negative coefficients c_{-n} ?
In DFT, they moved to the right end of the c_n vector :

$$c_{-n} = c_{N-n}$$

Fast Fourier transform (FFT)

Fast numerical realization of DFT is FFT. This is just the smart way to do DFT. Matlab has one built in

- y is a matlab vector of data points (y_k)
- `c=fft(y)` Fourier transform
- `y=ifft(c)` inverse Fourier transform

Notice that `fft` does not normalize by N. So to get Fourier series c_n, you need to calculate `fft(y)/N`.
However `y = ifft( fft(y) )`
Notice one more point of confusion: Matlab does not have index=0, so actual $c_n = c_{matlab\ fft}(n-1)$, so $c_0 = c_{matlab\ fft}(1)$