System of linear algebraic equations

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Lecture 21
Mobile problem

Someone provided us with 6 weights and 3 rods. We need to calculate the positions of suspension points to have a balanced system.

If the system is in equilibrium, torque must be zero at every pivot point

\[
\begin{align*}
w_1 x_1 - (w_2 + w_3 + w_4 + w_5 + w_6)x_2 &= 0 \\
w_3 x_3 - (w_4 + w_5 + w_6)x_4 &= 0 \\
w_5 x_5 - w_6 x_6 &= 0
\end{align*}
\]

We need 3 more equations. Let’s constrain the length of every rod

\[
\begin{align*}
x_1 + x_2 &= L_{12} \\
x_3 + x_4 &= L_{34} \\
x_5 + x_6 &= L_{56}
\end{align*}
\]
Mobile problem continued

Let’s define $w_{26} = w_2 + w_3 + w_4 + w_5 + w_6$ and $w_{46} = w_4 + w_5 + w_6$

$$w_1 x_1 - w_{26} x_2 = 0$$
$$w_3 x_3 - w_{46} x_4 = 0$$
$$w_5 x_5 - w_6 x_6 = 0$$

$$x_1 + x_2 = L_{12}$$
$$x_3 + x_4 = L_{34}$$
$$x_5 + x_6 = L_{56}$$

$$\sum_j A_{ij} x_j = B_i \rightarrow Ax = B$$

Matlab has a lot of built-in functions to solve problems in this form
The inverse matrix method

\[ Ax = B \]

\[ A^{-1}Ax = x = A^{-1}B \]

**Analytical solution**

\[ x = A^{-1}B, \quad \text{only if } \det(A) \neq 0 \]

**Matlab’s straightforward implementation (not the fastest)**

\[ x = \text{inv}(A) \times B \]

**Matlab’s faster way with the left division operator (recommended)**

\[ x = A \backslash B \]
Recall the mobile problem

If \( w_1 = 20, w_2 = 5, w_3 = 3, w_4 = 7, w_5 = 2, w_6 = 3, L_{12} = 2, L_{34} = 1, \) and \( L_{56} = 3, \) then \( w_{26} = 20 \) and \( w_{46} = 12. \)

\[
\begin{pmatrix}
20 & -20 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & -12 & 0 & 0 \\
0 & 0 & 0 & 2 & -3 \\
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6
\end{pmatrix}
=
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
2 \\
1
\end{pmatrix}
\]
Matlab mobile solution

A = [
  20, -20, 0, 0, 0, 0; ...
  0, 0, 3, -12, 0, 0; ...
  0, 0, 0, 0, 2, -3; ...
  1, 1, 0, 0, 0, 0; ...
  0, 0, 1, 1, 0, 0; ...
  0, 0, 0, 0, 1, 1; ...
]

B = [ 0; 0; 0; 2; 1; 3 ]

% 1st method
x = inv(A) * B

% 2nd method
x = A \ B

Check

>> A * x - B
0
0
0
0
0.2220
0

x =
1.0000
1.0000
0.8000
0.2000
1.8000
1.2000
To do or not to do the inverse matrix calculation

Solutions based on the inverse matrix calculation involve extra steps (unnecessary for solution) and, thus, are slower

```matlab
>> A=rand(4000);
>> B=rand(4000,1);
>> tic; x=inv(A)*B; toc
Elapsed time is 54.831124 seconds.
>> tic; x=A\B; toc
Elapsed time is 19.822778 seconds.
```

However, it is handy to calculate the inverse matrix in advance if you solve $Ax = B$ for different $B$ with the same $A$.

```matlab
>> tic; Ainv=inv(A); toc
Elapsed time is 58.304244 seconds.
>> B1=rand(4000,1); tic; x1=Ainv*B1; toc
Elapsed time is 0.048547 seconds.
>> B2=rand(4000,1); tic; x2=Ainv*B2; toc
Elapsed time is 0.048315 seconds.
```
Find the equivalent resistance of the following combination of resistors.
Wheatstone bridge problem

Find the equivalent resistance of the following combination of resistors.

\[ R_{eq} = \frac{V_b}{I_6} \]
%% Wheatstone bridge calculations
R1=1e3; R2=1e3; R3=2e3; R4=2e3; R5=10e3;
Vb=9;
A=[
    -1, -1, 0, 0, 0, 1; % I1+I2=I6 eq1
    1, 0, -1, 0, 1, 0; % I1+I5=I3 eq2
    0, 1, 0, -1, -1, 0; % I4+I5=I2 eq3
    0, 0, 1, 1, 0, -1; % I3_i4=I6 eq4
] % above would make a linear combination
% of the following  eq1+eq2=-(e3+eq4)
0, 0, R3, -R4, R5, 0; % R3*I3+R5*I5=R4*I4 eq4a
R1, 0, R3, 0, 0, 0; % R1*I1+R3*I3=Vb eq5
-R1, R2, 0, 0, R5, 0 % R2*I2+R5*I5=R1*I1 eq6
]
B=[0; 0; 0; 0; Vb; 0];

% Find currents
I=A\B

% equivalent resistance of the Wheatstone bridge
Req=Vb/I(6)