System of linear algebraic equations

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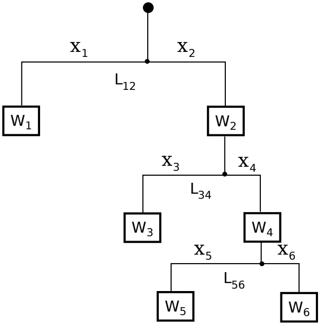
Lecture 21

Mobile problem

Someone provided us with 6 weights and 3 rods. We need to calculate the positions of suspension points to have a balanced system.

If the system is in equilibrium, torque must be zero at every pivot point

$$\begin{aligned} w_1 x_1 - (w_2 + w_3 + w_4 + w_5 + w_6) x_2 &= 0 \\ w_3 x_3 - (w_4 + w_5 + w_6) x_4 &= 0 \\ w_5 x_5 - w_6 x_6 &= 0 \end{aligned}$$



We need 3 more equations. Let's constrain the length of every rod

$$\begin{aligned} x_1 + x_2 &= L_{12} \\ x_3 + x_4 &= L_{34} \\ x_5 + x_6 &= L_{56} \end{aligned}$$

Mobile problem continued

Let's define $w_{26} = w_2 + w_3 + w_4 + w_5 + w_6$ and $w_{46} = w_4 + w_5 + w_6$

$$\begin{aligned} w_1 x_1 - w_{26} x_2 &= 0 \\ w_3 x_3 - w_{46} x_4 &= 0 \\ w_5 x_5 - w_6 x_6 &= 0 \\ x_1 + x_2 &= L_{12} \\ x_3 + x_4 &= L_{34} \\ x_5 + x_6 &= L_{56} \end{aligned}$$

$$\sum_j A_{ij} x_j = B_i \rightarrow \mathbf{Ax} = \mathbf{B}$$

Matlab has a lot of built-in functions to solve problems in this form

$$\begin{pmatrix} w_1 & -w_{26} & 0 & 0 & 0 & 0 \\ 0 & 0 & w_3 & -w_{46} & 0 & 0 \\ 0 & 0 & 0 & 0 & w_5 & -w_6 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ L_{12} \\ L_{34} \\ L_{56} \end{pmatrix}$$

The inverse matrix method

$$\mathbf{A}\mathbf{x} = \mathbf{B}$$

$$\mathbf{A}^{-1}\mathbf{A}\mathbf{x} = \mathbf{x} = \mathbf{A}^{-1}\mathbf{B}$$

Analytical solution

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{B}, \quad \text{only if } \texttt{det}(\mathbf{A}) \neq 0$$

Matlab's straight forward implementation (not the fastest)

$$\mathbf{x} = \texttt{inv}(\mathbf{A}) * \mathbf{B}$$

Matlab's faster way with the left division operator (recommended)

$$\mathbf{x} = \mathbf{A} \backslash \mathbf{B}$$

Recall the mobile problem

If $w_1 = 20$, $w_2 = 5$, $w_3 = 3$, $w_4 = 7$, $w_5 = 2$, $w_6 = 3$, $L_{12} = 2$, $L_{34} = 1$, and $L_{56} = 3$, then $w_{26} = 20$ and $w_{46} = 12$.

$$\begin{pmatrix} 20 & -20 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & -12 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -3 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 1 \\ 3 \end{pmatrix}$$

Matlab mobile solution

```
A=[ ...
20, -20,  0,   0,  0,   0; ...
 0,   0,  3, -12,  0,   0; ...
 0,   0,  0,   0,  2,  -3; ...
 1,   1,  0,   0,  0,   0; ...
 0,   0,  1,   1,  0,   0; ...
 0,   0,  0,   0,  1,   1; ...
 ]
B= [ 0; 0; 0; 2; 1; 3 ]
% 1st method
x=inv(A)*B
% 2nd method
x=A\B
```

```
x =
   1.0000
   1.0000
   0.8000
   0.2000
   1.8000
   1.2000
```

Check

```
>> A*x-B
   1.0e-15 *
     0
     0
     0
     0
     0.2220
     0
```

To do or not to do the inverse matrix calculation

Solutions based on the inverse matrix calculation involve extra steps (unnecessary for solution) and, thus, are slower

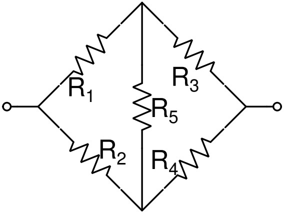
```
>> A=rand(4000);
>> B=rand(4000,1);
>> tic;  x=inv(A)*B; toc
Elapsed time is 54.831124 seconds.
>> tic;  x=A\B; toc
Elapsed time is 19.822778 seconds.
```

However, it is handy to calculate the inverse matrix in advance if you solve $\mathbf{Ax} = \mathbf{B}$ for different **B** with the same **A**.

```
>> tic; Ainv=inv(A);  toc
Elapsed time is 58.304244 seconds.
>> B1=rand(4000,1); tic; x1=Ainv*B1; toc
Elapsed time is 0.048547 seconds.
>> B2=rand(4000,1); tic; x2=Ainv*B2; toc
Elapsed time is 0.048315 seconds.
```

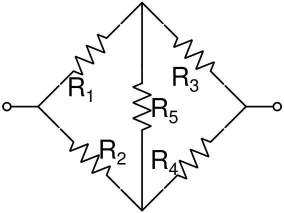
Wheatstone bridge problem

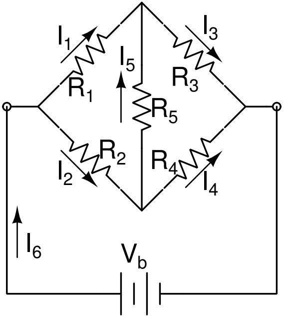
Find the equivalent resistance of the following combination of resistors.



Wheatstone bridge problem

Find the equivalent resistance of the following combination of resistors.





$$R_{eq} = \frac{V_b}{I_6}$$

Wheatstone bridge problem solution

```
%% Wheatstone bridge calculations
R1=1e3; R2=1e3; R3=2e3; R4=2e3; R5=10e3;
Vb=9;
A=[
  -1, -1,  0,   0,  0,  1;  % I1+I2=I6  eq1
   1,  0, -1,   0,  1,  0;  % I1+I5=I3  eq2
   0,  1,  0,  -1, -1,  0;  % I4+I5=I2  eq3
%  0,  0,  1,   1,  0, -1;  % I3_I4=I6  eq4
% above would make a  linear combination
% of the following  eq1+eq2=-(e3+eq4)
   0,  0, R3, -R4, R5,  0;  % R3*I3+R5*I5=R4*I4 eq4a
  R1,  0, R3,   0,  0,  0;  % R1*I1+R3*I3=Vb    eq5
 -R1, R2,  0,   0, R5,  0   % R2*I2+R5*I5=R1*I1 eq6
]
B=[0; 0; 0; 0; Vb; 0];

% Find currents
I=A\B

% equivalent resistance of the Wheatstone bridge
Req=Vb/I(6)
```