

A note on error propagation

We are trying to find the change (uncertainty) in some function $f(x, y, z)$ when x, y and z are subject to change (i.e., have some uncertainty). Assuming first that only x has an uncertainty δx and that our measured values (distinct from the variables x, y and z) are x_0, y_0, z_0 , we have:

$$|\delta f(x_0, y_0, z_0)| = |f(x_0 + \delta x, y_0, z_0) - f(x_0, y_0, z_0)| \quad (1)$$

$$\approx \left| \left[f(x_0, y_0, z_0) + \left. \frac{\partial f}{\partial x} \right|_{x=x_0} (x_0 + \delta x - x_0) \right] - f(x_0, y_0, z_0) \right| \quad (2)$$

$$= \left| \left. \frac{\partial f}{\partial x} \right|_{x=x_0} (\delta x) \right| \quad (3)$$

For uncorrelated uncertainties on x, y and z , this leads to the general formula:

$$(\delta f)^2 = \left(\frac{\partial f}{\partial x} \right)^2 (\delta x)^2 + \left(\frac{\partial f}{\partial y} \right)^2 (\delta y)^2 + \left(\frac{\partial f}{\partial z} \right)^2 (\delta z)^2 \quad (4)$$

Where we evaluate $f()$ and it's derivatives at (x_0, y_0, z_0) .

Example from the e/m lab

We'd like to propagate errors on e/m . First we write down a function for e/m in terms of the measured variables:

$$\frac{e}{m} = \left[\frac{125}{32} \frac{a^2}{\mu_0^2 N^2} \right] \frac{V}{I^2 r^2} = C \frac{V}{I^2 r^2} \quad (5)$$

where C is a constant and V, I, r are the measured variables. We can now propagate errors and find the uncertainty $(e/m)_j$, the measured value of e/m for trial j . We set $e/m = f(V, I, r)$ and then $(e/m)_j = f(V_j, I_j, r_j) \equiv f_j$. The uncertainty in f_j is:

$$(\delta f_j)^2 = \left(\left. \frac{\partial f}{\partial V} \right|_j \right)^2 (\delta V)^2 + \left(\left. \frac{\partial f}{\partial I} \right|_j \right)^2 (\delta I)^2 + \left(\left. \frac{\partial f}{\partial r} \right|_j \right)^2 (\delta r)^2 \quad (6)$$

$$= \left(\frac{C}{I_j^2 r_j^2} \right)^2 (\delta V_j)^2 + \left(\frac{-2CV_j}{I_j^3 r_j^2} \right)^2 (\delta I_j)^2 + \left(\frac{-2CV_j}{I_j^2 r_j^3} \right)^2 (\delta r_j)^2 \quad (7)$$

now we can rearrange a little bit

$$= \left(\frac{CV_j}{I_j^2 r_j^2} \right)^2 \left(\frac{\delta V_j}{V_j} \right)^2 + \left(\frac{CV_j}{I_j^2 r_j^2} \right)^2 \left(2 \frac{\delta I_j}{I_j} \right)^2 + \left(\frac{CV_j}{I_j^2 r_j^2} \right)^2 \left(2 \frac{\delta r_j}{r_j} \right)^2 \quad (8)$$

But $\left(\frac{CV_j}{I_j^2 r_j^2}\right)^2 = (f_j)^2$, so we have

$$(\delta f_j)^2 = (f_j)^2 \left(\frac{\delta V_j}{V_j}\right)^2 + (f_j)^2 \left(2\frac{\delta I_j}{I_j}\right)^2 + (f_j)^2 \left(2\frac{\delta r_j}{r_j}\right)^2 \quad (9)$$

The terms $\frac{\delta V_j}{V_j}$, $\frac{\delta I_j}{I_j}$, $\frac{\delta r_j}{r_j}$ are just fractional uncertainties in the measured voltage, current and radius. In this case, you had something like $\frac{\delta V_j}{V_j} \approx \frac{0.05 \text{ V}}{250 \text{ V}} = 0.02\%$, something similar for I , but $\frac{\delta r_j}{r_j} \approx \frac{0.3 \text{ cm}}{4.0 \text{ cm}} = 7.5\%$, so the last term in Eq. 9 completely dominates the other two. Therefore:

$$|\delta f_j| \approx |f_j| \left| 2\frac{\delta r_j}{r_j} \right| = 15\%|f_j| \quad (10)$$

This is the sort of analysis I'd like to see in a report and Eq. 10 is what you'd code into an Igor or Excel spreadsheet. Of course Eq. 9 would also work, you'd just miss out on some intuition about what was driving the uncertainties in your experiment and could be more vulnerable to typos in the formula.

A slightly different approach

What if derivatives of f are messy to evaluate? For example as in the electron diffraction lab when you are trying to propagate uncertainties on $\sin \theta$ which involves a double angle formula, sines and inverse sines of the quantities you measured. There is a simple thing you can do, going back to Eq. 1:

$$|\delta f(x_0, y_0, z_0)| = |f(x_0 + \delta x, y_0, z_0) - f(x_0, y_0, z_0)| \quad (11)$$

which can be generalized to multiple dimensions as:

$$\begin{aligned} (\delta f(x_0, y_0, z_0))^2 &= [f(x_0 + \delta x, y_0, z_0) - f(x_0, y_0, z_0)]^2 \\ &+ [f(x_0, y_0 + \delta y, z_0) - f(x_0, y_0, z_0)]^2 \\ &+ [f(x_0, y_0, z_0 + \delta z) - f(x_0, y_0, z_0)]^2 \end{aligned}$$

That is, we just reevaluate f when we shift x_0, y_0, z_0 . The shifts are independent so long as the uncertainties themselves are independent, and we add all shifts in quadrature. This avoids taking derivatives and may, or may not, be less messy. Also, note, I claim it's a little better to do this:

$$(\delta f(x_0, y_0, z_0))^2 = \left[\frac{1}{2} |f(x_0 + \delta x, y_0, z_0) - f(x_0, y_0, z_0)| \right. \quad (12)$$

$$+ \left. \frac{1}{2} |f(x_0 - \delta x, y_0, z_0) - f(x_0, y_0, z_0)| \right]^2$$

$$+ \quad \text{similar terms for } y \text{ and } z \quad (13)$$

That is, average the effect of shifting by $\pm\delta x$.

Final Note

Whatever you do, be clear about what you are assuming for input uncertainties and the specific formula for δf ! We don't need to see all steps in deriving δf but we must be able to plug your x_i, y_i, z_i and $\delta x_i, \delta y_i, \delta z_i$ into your formula to get your δf_i .