Lecture 21 Stellar pulsations

Cepheids. — Pulsating (changing luminocity stars)

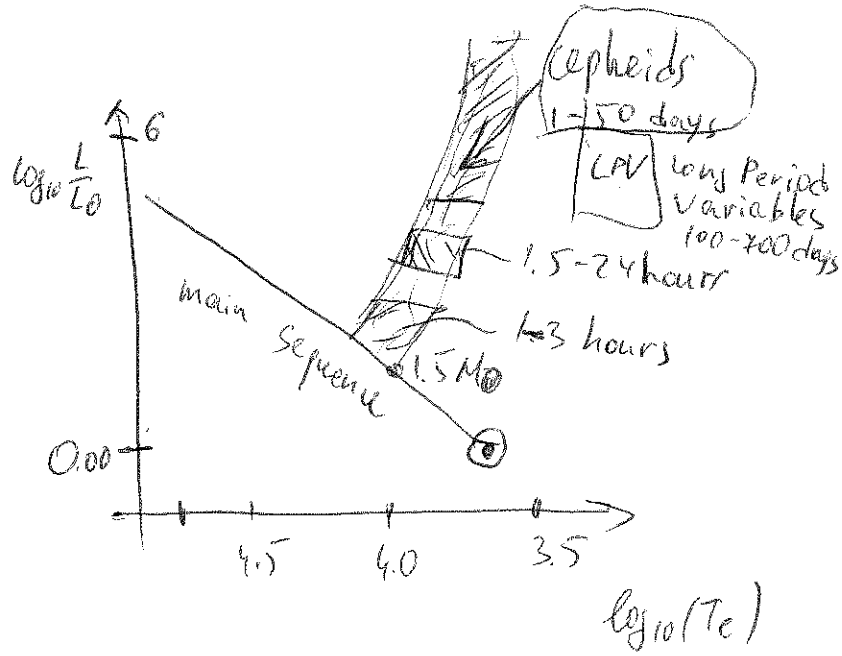
By 2005 - 40K of such stars discovered with periods from hours → to days → years

Q: What is a big deal about them?
A: there is one to one relationship $L \Longleftrightarrow$ Period.

Thus if we know P, we know L, and then we now how far is such star away from us.
Cepheids are "standard candles"

$$\log_{10} \frac{\langle L \rangle}{L_\odot} = 1.15 \log_{10} P_d + 2.47$$

Instability strip



Speed of sound

$$v_s = \sqrt{\frac{\gamma P}{\rho}} \quad , \quad \gamma = \frac{c_p}{c_v} = \frac{5}{3} \text{ for ideal gas}$$

Recall hydrostatic equilibrium condition
assuming $\rho = \text{const}$

$$\frac{dP}{dr} = -g\rho = -\frac{G M_r}{r^2}\rho = -\frac{G}{r^2}\rho\frac{4\pi}{3}r^3\rho$$

$$= -\frac{4}{3}\pi G r \rho^2$$

$$\Downarrow$$

$$P(r) = \frac{2}{3}\pi G \rho^2 (R^2 - r^2)$$

$$\text{period } \Pi = 2\int_0^R \frac{dr}{v_s} =$$

Q: why

~~test~~ to center and back

$$= 2\int \frac{dr}{\sqrt{\frac{2}{3}\gamma\pi G\rho(R^2 - r^2)}} =$$

$$= \frac{2\sqrt{\frac{3}{2}}}{\sqrt{\gamma\pi G\rho}} \underbrace{\int_0^R \frac{dr}{\sqrt{R^2 - r^2}}}_{//\ x = r/R} = \frac{2\sqrt{\frac{3}{2}}}{\sqrt{\gamma\pi G\rho}} \cdot \underbrace{\operatorname{atan}\left(\frac{r}{\sqrt{R^2 - r^2}}\right)\Bigg|_0^R}_{\left(\frac{\pi}{2} - 0\right)}$$

$$\int \frac{dx}{\sqrt{1 - x^2}} = /x = \cos\theta/ = \int_{\pi/2}^{0} -\frac{\sin\theta\, d\theta}{\sin\theta} = \pi/2$$

$$\Pi = \sqrt{\frac{3\pi}{2\gamma G\rho}}$$

Typical Cepheid $M = 5 M_\odot$
$R = 50 R_\odot$

$$\boxed{\Pi = \sqrt{\frac{3}{2}\frac{\pi}{\gamma G \rho}} = 2.06 \cdot 10^5 \cdot \frac{1}{\sqrt{\rho}}}$$

For sun $\Pi = 2.06 \cdot 10^5 \cdot \frac{1}{\sqrt{1400}} = 5500$ sec $= 1.5$ hours
not observed

A more realistic cepheid

$M = 5 M_\odot$, $R = 50 R_\odot$

$$\rho = \frac{M}{\frac{4\pi}{3} R^3} = \frac{5 \overbrace{M_\odot}^{=\rho_\odot}}{\frac{4\pi}{3} R_\odot^3 \cdot (50)^3} = \frac{5}{50^3}\rho_\odot =$$

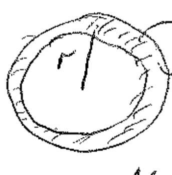
$$= 4 \cdot 10^{-5} \rho_\odot$$

$$\boxed{\Pi_{cepheid} = \Pi_\odot \frac{1}{\sqrt{4 \cdot 10^{-5}}} = 8.7 \cdot 10^5 \approx 10 \text{ days}}$$

More accurate hydrodynamic model

Recall $$\rho \frac{d^2 r}{dt^2} = -G\frac{M_r \rho}{r^2} - \frac{dP}{dr} \qquad (\text{eq } 1)$$

We do thin ~~outside~~ outer shell model



$V = 4\pi r^2 dr$

$m = \rho V$

Multiply eq. 1 by volume of the shell we will arrive to simple Newtons law

$$\underbrace{4\pi r^2 dr \cdot \rho}_{m} \frac{d^2 r}{dt^2} = -G\frac{M_r}{r^2}\underbrace{\rho 4\pi r^2 dr}_{m} - 4\pi r^2 \cancel{dr}\frac{dP}{\cancel{dr}}$$

$$m\frac{d^2 r}{dt^2} = -G\frac{M_r m}{r^2} - 4\pi r^2 dP$$

this $\underbrace{P_{outside}}_{0} - \underbrace{P_{inside}}_{P}$

$$m\frac{d^2 r}{dt^2} = -G\frac{M_r m}{r^2} + 4\pi r^2 P$$

notice sign change

assuming R_0 and P_0 equilibrium values

i.e.

$$\boxed{0 = -\frac{G M_r m}{R_0^2} + 4\pi R_0^2 P_0}$$

assuming that our shell do small perturbations aroun equilibrium

constants

$$r = R_0 + \delta R \quad , \quad P = P_0 + \delta P$$

$$\frac{m\, d^2(R_0+\delta R)}{dt^2} = \frac{m d^2 \delta R}{dt^2} = -\frac{G M_r m}{(R_0+\delta R)^2} + 4\pi (R_0+\delta R)^2 (P_0+\delta P)$$

We do Taylor expansion

$$(R_0 + \delta R)^\alpha = \left(R_0\left(1+\frac{\delta R}{R_0}\right)\right)^\alpha =$$

$$\approx R_0^\alpha\left(1+\alpha\frac{\delta R}{R_0}\right) \quad \text{and drop terms} \sim \left(\frac{\delta R}{R_0}\right)^2 \text{ and higher powers}$$

$$m\frac{d^2\delta R}{dt^2} \approx -\frac{G M_r m}{R_0^2}\left(1-2\frac{\delta R}{R_0}\right) + 4\pi R_0^2\left(1+2\frac{\delta R}{R_0}\right)(P_0+\delta P)$$

(see equilibrium eq)

$$= \cancel{-\frac{G M_r m}{R_0^2}} + 2\frac{G M_r m}{R_0^2}\frac{\delta R}{R_0} + \cancel{4\pi R_0^2 P_0} +$$

$$+ 4\pi R_0^2\, 2\frac{\delta R}{R_0} P_0 + 4\pi R_0^2 \delta P \frac{P_0}{P_0} + \cancel{4\pi R_0^2\, 2\frac{\delta R}{R_0}(\delta P)}$$

$0 \sim \delta R \cdot \delta P$ we keep only linear terms

$$= \Big/ \text{Notice from equilibrium } 4\pi R_0^2 P_0 = \frac{G M_r m}{R_0^2} \Big/$$

$$\frac{\cancel{m}\, d^2\delta R}{dt^2} = \frac{G M_r \cancel{m}}{R_0^2}\left[4\left(\frac{\delta R}{R_0}\right) + \frac{\delta P}{P_0}\right] \qquad \text{(eq. 2)}$$

Adiabatic process - no heat in or out

i.e. change of internal energy only due to work on gas

$$\Delta U = W = -\int p\,dV$$

$U = \frac{f}{2} NkT$

$f \equiv$ # degrees of freedom

$$\frac{f}{2} Nk\Delta T = \frac{f}{2} Nk(T_2 - T_1) = -\int p\,dV$$

$$\frac{f}{2} Nk\int dT = -\int p\,dV$$

ideal gas $pV = NkT$

$$\Rightarrow \quad \frac{f}{2} Nk\,dT = -p\,dV$$

$$\frac{f}{2} \cancel{Nk}\,dT = -\frac{\cancel{Nk}T}{V}\,dV$$

$$\frac{f}{2} \frac{dT}{T} = -\frac{dV}{V}$$

$$\frac{f}{2} \ln T \Big|_{T_1}^{T_2} = -\ln V \Big|_{V_1}^{V_2}$$

$$\frac{f}{2} \ln T_2/T_1 = -\ln V_2/V_1 = \ln \frac{V_1}{V_2}$$

$$\left(\frac{T_2}{T_1}\right)^{f/2} = \frac{V_1}{V_2} \Rightarrow \boxed{\boxed{T^{f/2} V} = \text{const}}$$

$$T^{f/2} V = \left(\frac{pV}{Nk}\right)^{f/2} V \Rightarrow pV \cdot V^{2/f}$$

const so we drop it

$$\Rightarrow \quad pV^{\frac{f+2}{f} = \gamma} = \boxed{pV^{\gamma} = \text{const}}$$

revisit $T^{f/2} V \Rightarrow TV^{2/f} = \boxed{TV^{\gamma - 1} = \text{const}}$

We need to move from $\frac{\delta P}{P_0}$ to δR

Adiabatic (no energy exchange) ~~exp~~ expansion / contraction: $PV^{\gamma} = const$

$$PV^{\gamma} = const \Rightarrow P(R^3)^{\gamma} = PR^{3\gamma} = const_2$$

$$PR^{3\gamma} = (P_0 + \delta P)(R_0 + \delta R)^{3\gamma} =$$

$$= (P_0 + \delta P)\left[R_0\left(1 + \frac{\delta R}{R_0}\right)\right]^{3\gamma} \simeq (P_0 + \delta P) R_0^{3\gamma}\left(1 + 3\gamma\frac{\delta R}{R_0}\right)$$

$$= \underbrace{P_0 R_0^{3\gamma}}_{= const_2} + \underbrace{P_0 R_0^{3\gamma}\left(3\gamma\frac{\delta R}{R_0}\right) + \delta P R_0^{3\gamma}}_{\text{must be equal to 0 so RHS} = const_2} + \dots \underbrace{\cancel{\delta R \delta P}}_{\text{too small}}$$

$$P_0 \cancel{R_0^{3\gamma}}\left(3\gamma\frac{\delta R}{R_0}\right) = -\delta P \cancel{R_0^{3\gamma}}$$

$$\boxed{\frac{\delta P}{P_0} = -3\gamma\frac{\delta R}{R_0}}$$

let's plug it to eq ?

$$\frac{d^2 \delta R}{dt^2} = \frac{G M_r}{R_0^2}\left[4\left(\frac{\delta R}{R_0}\right) + \frac{\delta P}{P_0}\right] =$$

$$= \frac{G M_r}{R_0^2}\left[4\,\frac{\delta R}{R_0} - 3\gamma\,\frac{\delta R}{R_0}\right]$$

let's ~~drop~~ $M_r \to M$

$$(\ddot{\delta R}) = -\left[3\gamma - 4\right]\frac{GM}{R_0^3}\,\delta R$$

$$\Rightarrow \quad \delta R = A\cos(\omega t) + B\sin(\omega t)$$

i.e. periodic

$$\omega^2 = (3\gamma - 4)\frac{GM}{R_0^3}$$

$$\Pi = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{(3\gamma-4)\frac{GM}{R_0^3}\cdot\frac{\frac{4\pi}{3}}{\frac{4\pi}{3}}}} = \frac{2\pi}{\sqrt{\frac{4\pi}{3}(3\gamma-4)G\rho}}$$

$$= \boxed{\sqrt{\frac{3\pi}{(3\gamma-4)G\rho}} = \Pi}$$

~~sound speed~~ pressure is restoring mechanism

Compare it to simple round trip estimate earlier

$$\Pi_{simple} \simeq \sqrt{\frac{3\pi}{2\gamma G\rho}}$$

same except some numeric factor

Well, it all very nice to see oscillations of star size but we detect luminocity. So what about L ?

Recall, $L = 4\pi R^2 \sigma T^4$

$$dL = 4\pi R \cdot 2dR\, \sigma T^4 + 4\pi R^2 \sigma\, 4T^3 dT$$

$$= L_0 \frac{2dR}{R_0} + L_0 \frac{4dT}{T_0}$$

$$\frac{dL}{L_0} = \frac{2dR}{R_0} + \frac{4dT}{T_0}$$

Recall in adiabatic process $TV^{\gamma-1} = \text{const}$

$$\Rightarrow T R^{3(\gamma-1)} = \text{const}$$

$$dT\, R^{3(\gamma-1)} + T R^{3(\gamma-1)-1} dR \cdot 3(\gamma-1) = 0$$

$$\boxed{\frac{dT}{T} = -3(\gamma-1)\frac{dR}{R}}$$

$$\frac{dL}{L_0} = \frac{2dR}{R} + 4\cdot(-3(\gamma-1))\frac{dR}{R} =$$

$$= 2(1-6(\gamma-1))\frac{dR}{R}$$

for ideal gas $\gamma = 5/3$

So $$\frac{dL}{L_0} = 2(1-6\cdot\tfrac{2}{3})\frac{dR}{R} = -6\frac{dR}{R}$$

i.e. luminocity higher when star shrinks!