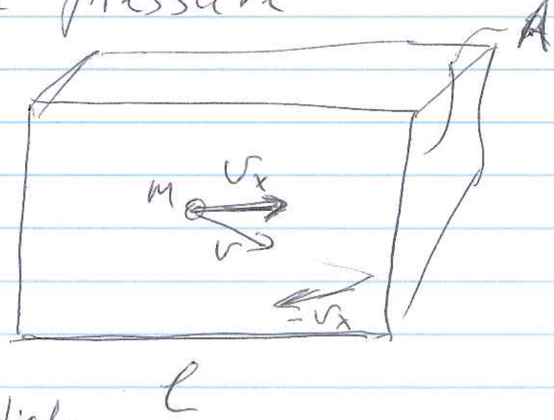
Lecture 18 $P \leftrightarrow T \leftrightarrow$ fusion

OK we know P but what about temperature?

We need to recall the origin of pressure



per particle

$$F = \frac{\Delta p}{\Delta t} = \frac{m v_x - m(-v_x)}{\Delta t} =$$

$$= \frac{2 m v_x}{(2\ell / v_x)} = \frac{m v_x^2}{\ell}$$

round trip time

$$v_x^2 + v_y^2 + v_z^2 = v$$

in average $v_x^2 = v_y^2 = v_z^2$

$$\Rightarrow v_x^2 = \frac{v^2}{3}$$

$$F = \frac{1}{3} \frac{m v^2}{\ell} = \text{recall the construction} = \frac{1}{3} \frac{p v}{\ell}$$

$$P_{\text{pressure per particle}} = \frac{F}{A} = \frac{1}{3} \frac{p v}{\ell \cdot A} = \frac{1}{3} \frac{p v}{V}$$

P2

$$\underset{\text{pressure}}{P} = \int_{\text{all particles}} \frac{1}{3} p v_0 \underset{\text{density}}{n_p} \, dp = \qquad \frac{N_p}{V}$$

momentum

$$= \frac{1}{3V}\int p v \, n_p \, dp$$

also $\int n_p \, dp = \frac{N}{V} = n$ $\quad \rho/m$

$$\boxed{P = \frac{1}{3}\int_0^\infty p v \, n_p \, dp}$$

For ideal gas we have Maxwell-Boltzmann distribution

$$n_v \, dv = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mv^2}{2kT}} 4\pi v^2 \, dv$$

note that $v = \frac{p}{m}$ for atoms so

$$\int n_v \, dv = \int n_p \, dp = \int n_p \, m \, dv$$

$$\Rightarrow n_p = \frac{n_v}{m} \; ; \quad m v^2 = \frac{p^2}{m}$$

Thus $\boxed{P_{gas} = nkT} = \boxed{\frac{\rho kT}{\bar{m}} = \frac{\rho kT}{\mu m_H}}$

MW

$\frac{N_p}{V} = \frac{\rho}{\bar{m}}$ $\quad$ 0.62 for sun

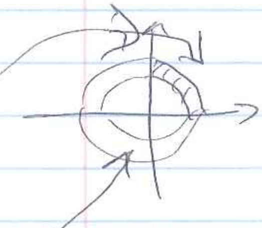
$N_p = N_H + N_{He} +$ other atoms

$M = N_H \cdot m_H + N_{He} \cdot m_{He} + \dots = N \cdot \mu \cdot m_H \Rightarrow \mu = \frac{\bar{m}}{m_H}$

mean molecular mass

helphul tricks to calculate
M-B integrals

$$\left(\int_0^\infty e^{-x^2}\, dx\right)\cdot\left(\int_0^\infty e^{-y^2}\, dy\right)$$



$$\iint_0^\infty e^{-r^2}\, dx\, dy = \int_0^\infty e^{-r^2}\, \frac{2\pi r\, dr}{4}$$

$$= \int_0^\infty e^{-r^2}\, \frac{\pi\, dr^2}{4} = \frac{\pi}{4} \Rightarrow \boxed{\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}}$$

$$\cancel{e^{-x^2} x^2\, dx}$$

$$\int_0^\infty d\left(e^{-x^2} x\right) = \int_0^\infty -e^{-x^2}\, 2x^2\, dx + \int_0^\infty e^{-x^2}\, dx$$

$\sqrt{\pi}/2$

$$0 = -2\int e^{-x^2} x^2\, dx + \boxed{\int e^{-x^2}\, dx}$$

$$\Rightarrow \boxed{\int e^{-x^2} x^2\, dx = \frac{\sqrt{\pi}}{4}}$$

P4

What about photon gas

$$P_{ph} = \frac{1}{3}\int_0^\infty p\, \overset{c}{v}\, n_p\, dp \quad (p = \frac{h\nu}{c}) = \frac{1}{3}\int h\nu\, n_p\, dp =$$

$\boxed{MW}$ $\longrightarrow$ $= \frac{1}{3}\int h\nu\, n_\nu\, d\nu = \frac{1}{3} u$

$$P_{rad} = \frac{1}{3} a T^4 \quad , \quad \boxed{a = \frac{4\sigma}{c}}$$ ← Boltzmann constant

$$\boxed{P = P_{Gas} + P_{rad} = \frac{\rho k T}{\mu m_H} + \frac{1}{3} a T^4}$$

Now we can estimate Sun temperature

~~μ~~ $\mu = 0.62$ ← looks like quite ionized gas

$$\frac{m_{H^+} + m_e}{m_H} \approx \frac{1}{2}$$

disregard Prad for now

$$\text{So } T_c = \frac{P_c \mu m_H}{\rho k} = \frac{4 \cdot 10^{14} \cdot 0.62 \cdot 1.67 \cdot 10^{-27}}{1.4 \cdot 1.38 \cdot 10^{-23}}$$

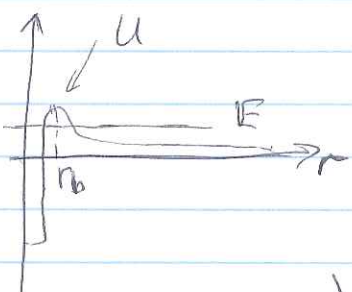
$= 2.1 \cdot 10^7\ K$

even if we use improved pressure model $P = 2.64 \cdot 10^{16}$ $T \approx 1.4 \cdot 10^9$ still order of magnitude low

Gramov peak

Ok. Clearly classical mechanics does not solve the problem of fusion. Sun is just not hot enough. since it requires 10^{10} K

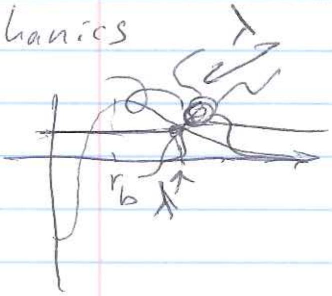
In average hydrogen atoms have energy much smaller ~~than~~ to penetrate the coulomb repulsion barrier



Quantum mechanics

$$\Delta X \Delta p_x \geq \hbar/2$$

$$\lambda = \frac{h}{p} \approx r_b$$



$$K_c \frac{Z_1 Z_2 e^2}{r_b} = K_c \frac{Z_1 Z_2 e^2}{\lambda} = K_c \frac{Z_1 Z_2 e^2}{\hbar / p} = E = \frac{p^2}{2\mu}$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$ reduced math

~~$\frac{p^2}{2m} = \frac{3}{2} kT \Rightarrow p = \sqrt{\frac{3}{2} 2mkT}$~~

$$K_c \frac{Z_1 Z_2 e^2}{\lambda} = \frac{(h/\lambda)^2}{2\mu}$$

$$\lambda = \frac{h^2}{2\mu K_c Z_1 Z_2 e^2} = \frac{(6.6 \cdot 10^{-34})^2}{2 \cdot \frac{1.6 \cdot 10^{-27}}{2} \cdot 9 \cdot 10^9 \cdot (1.6 \cdot 10^{-19})^2}$$

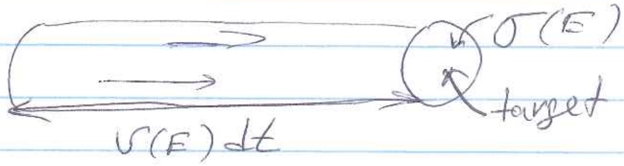
$$\mu = \frac{m_H}{2} \qquad = 1.2 \cdot 10^{-12}\ m$$

$$Z_1 = Z_2 = 1$$

much weaker (3 order) requirements compared to 10^{-15} m

$$\boxed{K_c \frac{Z_1 Z_2 e^2}{\lambda} = \frac{3}{2} kT}$$

$\boxed{\text{for fusion } 1.0^7 \text{ would be enough}}$



concentration

Number of collisions at given Energy per target $\quad n_{iE} \cdot \underbrace{\sigma(E)\, v(E)\, dt}_{\text{volume}}$

incident

$$dN_E = \sigma(E)\, v(E)\, n_{iE}\, dE\, dt$$

$$dE\, n_{iE} = \frac{n_i}{n} n_E\, dE$$

<u>reaction rate</u> r (collisions per unit of time)

$$r_{ix} = \int n_x n_i\, \sigma(E)\, v(E)\, \frac{n_E}{n}\, dE$$

Why book use x instead of t? $x \to +$ mnemonic

$$\sigma(E) \approx \lambda^2 = \left(\frac{h}{p}\right)^2 \sim \frac{1}{E}$$

strictly speaking not all collision lead to reactions. So the must be probability of reaction factored in which is comonly absorbed into σ. Such probability will be proportional to tunneling prob.

$$\sim e^{-2\pi^2 U_c/E}$$

Lecture 18 stops here