Lecture 6
Kepler's law and mass determination

$\alpha = G M_1 m_2$

$\mu = \frac{m_1 m_2}{m_1 + m_2}$

Recap

$$e = \sqrt{1 + \frac{2EL^2}{\mu \alpha^2}}$$

$$p = \frac{L^2}{\mu \alpha^2} \quad ,$$

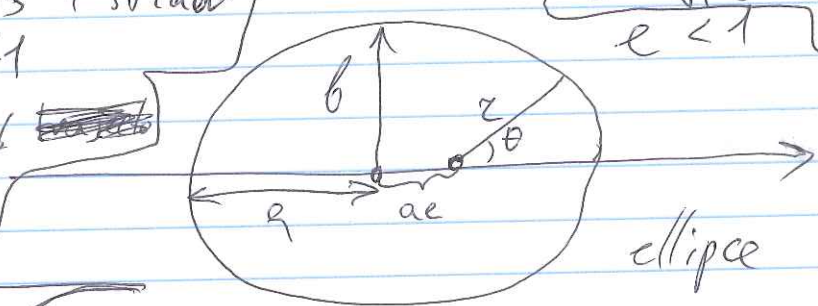
$$\boxed{\begin{aligned} \frac{p}{r} &= 1 + e\cos\theta \\ a &= \frac{p}{1-e^2} = \frac{\alpha}{2|E|} \\ b &= \frac{p}{\sqrt{1-e^2}} = \frac{L}{\sqrt{2\mu|E|}} \end{aligned}}$$

$e < 1$

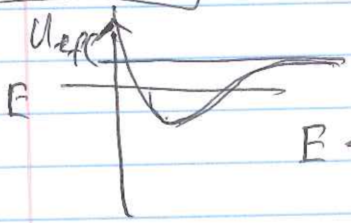
Kepler's 1st law

$e \leq 1$

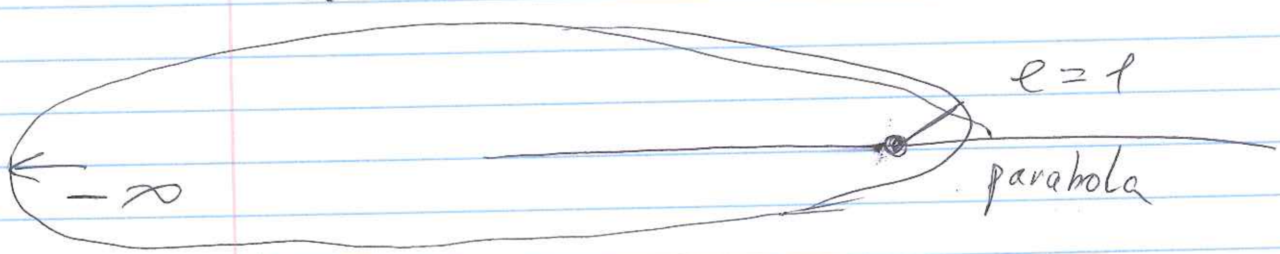
bound orbits



ellipce

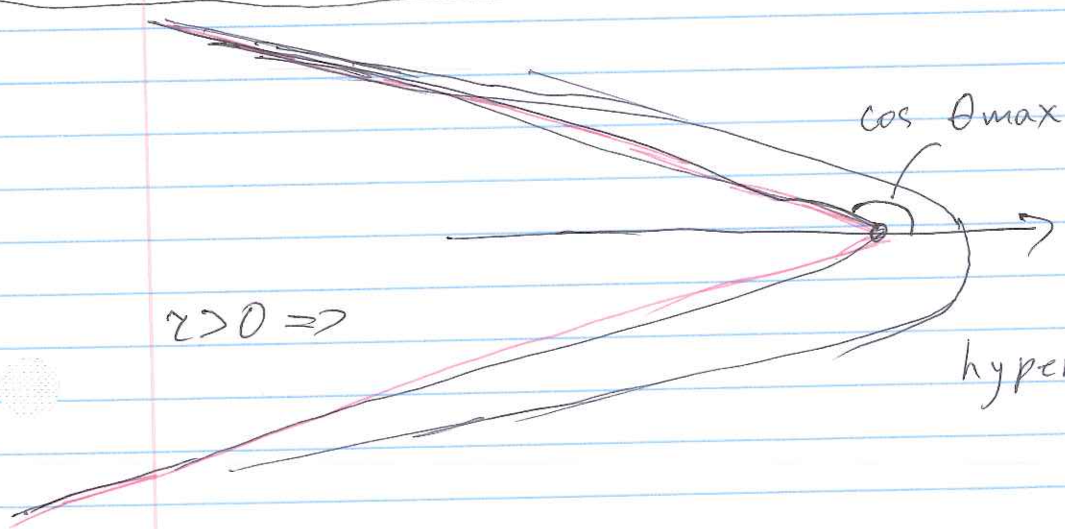


$E < 0 \Rightarrow e < 1$



$e = 1$

parabola



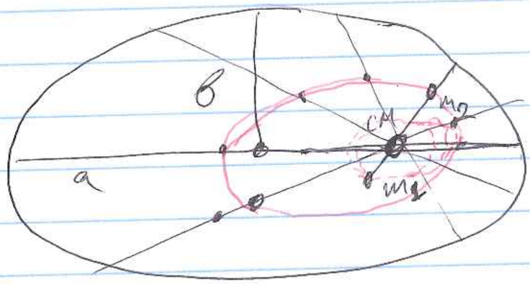
$\cos\theta_{max} = -\frac{1}{e}$

$e > 1$

$r > 0 \Rightarrow$

hyperbola

Let's talk about ellipces again



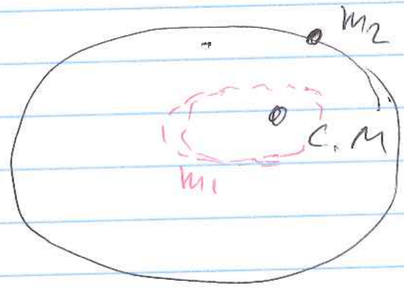
$\vec{r}_1 = -\frac{\mu}{m_1}\vec{r}$

$\vec{r}_2 = -\frac{\mu}{m_2}\vec{r}$

$\Rightarrow \boxed{a_1 + a_2 = a}$

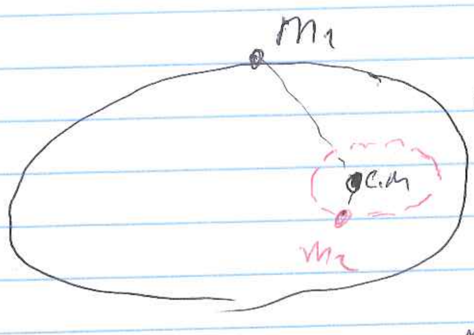
$\boxed{e_1 = e_2 = e}$

Before some Q



Is it correct? i.e. for both ellipces ~~...~~ long sites on the same side?

see above to see that this is Wrong

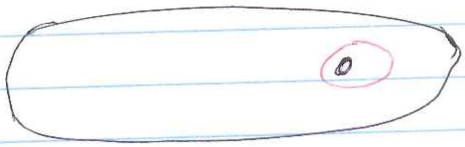


at some time t

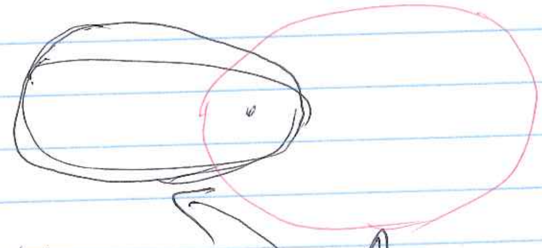
is it possible?

No! By construction m_1, C.M., m_2 are on the same line

Q: Which is correct



or

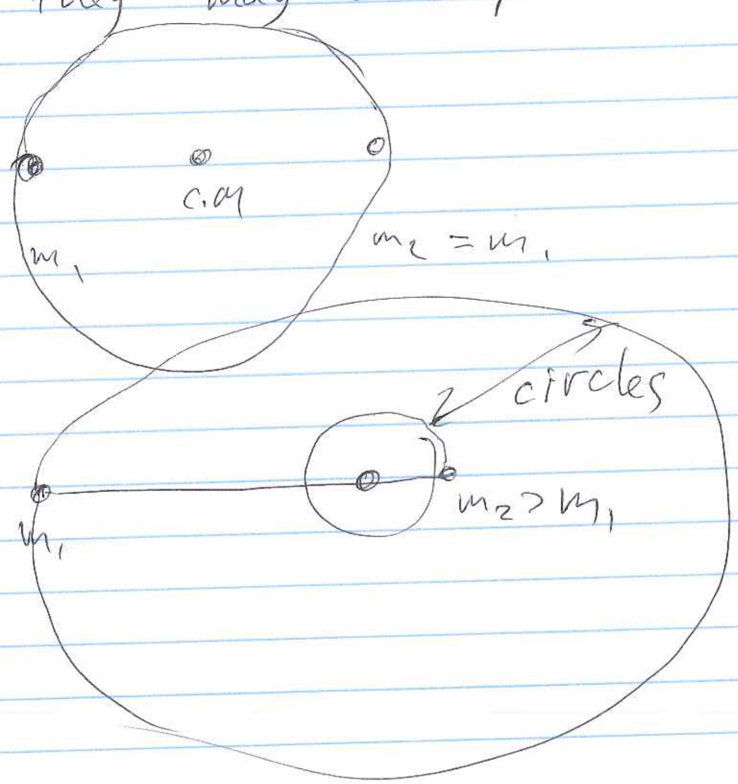


A: Both is possible

<u>Fun problem</u>: condition for intersection

Find $\frac{m_1}{m_2}$ and e which leads to

Some known case for $e = 0$ it is not possible. they may overlap but do not cross.



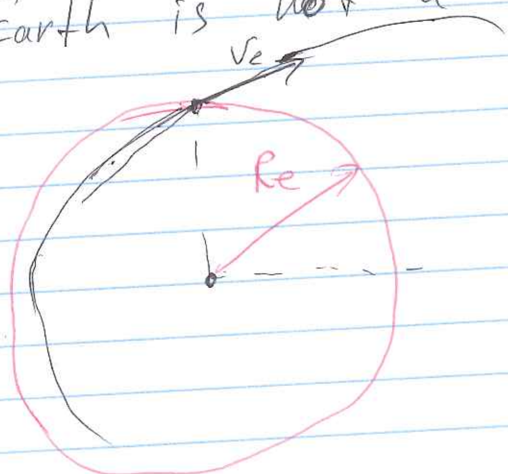
Q tricky: from gravity bound motion

$r > 0$ $\qquad \frac{p}{r} = 1 + e\cos\theta$

Why thrown stones fall on Earth?
i.e: r <u>seems</u> to be zero!

Trick replace r with $r - R_{Earth}$

Which is not the same $\Longleftrightarrow$ also Earth is not a point.

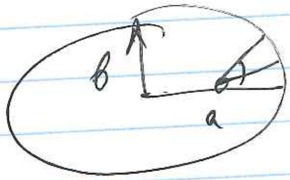


Find period:

$$A = \int_0^{P} \frac{dA}{dt} = \frac{L}{2\mu}\int_0^{P} dt =$$

← period

area

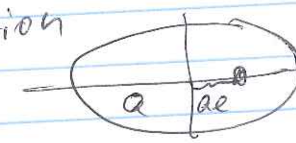


$$\pi a b = \frac{L}{2\mu} P$$

$$\pi \frac{p}{1-e^2}\cdot\frac{p}{\sqrt{1-e^2}} = \frac{L}{2\mu} P = \pi a^2\sqrt{1-e^2}$$

$$\frac{\pi}{(\sqrt{1-e^2})^3}\left(\frac{L^2}{\mu\alpha}\right)^2 = \frac{L}{2\mu} P = \pi a b = \pi a^2\sqrt{1-e^2}$$

let's find a from energy conservation



$$E = \frac{\mu(\dot r)^2}{2} + \frac{L^2}{2\mu r^2} + -\frac{\alpha}{r}$$

perehelion $\Leftrightarrow$ $\dot r = 0$ $\Leftrightarrow$ aphelion

$$E = \frac{L^2}{2\mu r_p^2} - \frac{\alpha}{r_p} = \frac{L^2}{2\mu r_a^2} - \frac{\alpha}{r_a}$$

$r_p = a(1-e)$; $r_a = a(1+e)$

$$\cancel{\frac{L^2}{2\mu a^2(1-e)^2} - \frac{\alpha}{a(1-e)} = \frac{L^2}{2\mu a^2(2-e)^2} - \frac{\alpha}{a(1+e)}}$$

$$E = \frac{L^2}{2\mu a^2(1-e)^2} - \frac{\alpha}{a(1-e)} = \frac{L^2}{2\mu a^2(1+e)^2} - \frac{\alpha}{a(1+e)}$$

$$\frac{L^2}{2\mu a}\left(\frac{1}{(1-e)^2} - \frac{1}{(1+e)^2}\right) = \alpha\left(\frac{1}{1-e} - \frac{1}{1+e}\right)$$

$$\frac{L^2}{\cancel{2}\mu a}\,\frac{\cancel{4}^{2}e}{[(1-e)(1+e)]^{\cancel{2}}} = \frac{\alpha\,\cancel{2e}}{\cancel{(1-e)(1+e)}}$$

$$L = \sqrt{a\mu\alpha(1-e^2)} =$$

$$= \sqrt{a \mu\, G\,\frac{m_1 m_2}{m_1+m_2}\,\frac{\overset{M}{(m_1+m_2)}}{1}(1-e^2)}$$

$$\boxed{L = \mu\sqrt{aGM(1-e^2)} = \sqrt{a\mu\alpha(1-e^2)}}$$

$$\pi a b = \pi a^2 \cancel{\sqrt{1-e^2}} = \frac{1}{2\mu} L P = \frac{1}{2\mu} P \sqrt{a \mu \alpha \cancel{(1-e^2)}}$$

$$P = 2\pi \, a^{3/2} \sqrt{\frac{\mu}{\alpha}}$$

$$P^2 = \frac{4\pi^2 \mu}{\alpha} a^3$$

$$P^2 = \frac{4\pi^2 \frac{m_1 m_2}{m_1 + m_2}}{G m_1 m_2} a^3$$

$$\boxed{P^2 = \frac{4\pi^2}{G(m_1 + m_2)} a^3}$$

Q: let's look at problem 2.14

Halley's comet has $P = 76$ year and $e = 0.9673$.

Can we find a, and ~~$M_\odot$~~ sun mass ($M_\odot$), and comet mass (m_c) from this information alone?

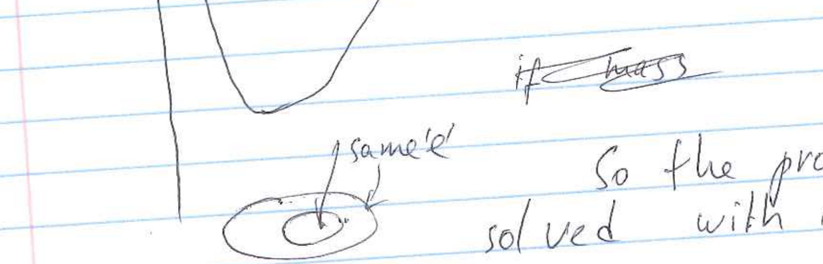
Lecture 6 Stops Here

$\boxed{\text{Lecture 7}}$

Well we know $P^2 = \frac{4\pi^2}{G M_{Total}} a^3$

we need M_T and a, so we need at least 1 more eq., so we can use one for 'e'.

So sounds like easy problem.



But 'e' does not bound anything ~~for~~ we can have different 'a's with the same e,

So the problem cannot be solved with only provided inform

So we need to use the fact that we know something else about solar system i.e Earth has $P = 1$ year and $a = 1AU$ $M_\odot + M_\oplus \simeq M_\odot + \ldots$