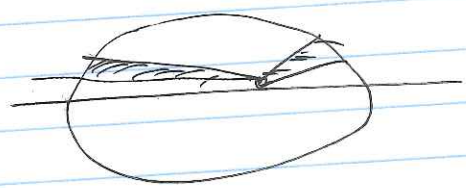
Lecture 5

Kepler's law derivations

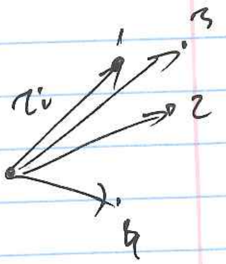
1. Planets move along ellipces, with Sun in focus
2. Planet covers the same area per unit time



3. $P^2 = a^3$, a is average distance in A.U
P is in years

total angular momentum conservation

$$\vec{L} = \sum m_i \vec{v}_i \times \vec{r}_i$$



in closed system where particles act only on themselve

$$\frac{d\vec{L}}{dt} = \sum m_i \left[\dot{\vec{v}}_i \times \vec{r}_i + \underbrace{\vec{v}_i \times \dot{\vec{r}}_i}_{\vec{v}_i \times \vec{v}_i = 0} \right] =$$

$$= \underbrace{m_i \dot{\vec{v}}_i}_{F_i} \times \vec{r}_i = \sum \vec{F}_i \times \vec{r}_i =$$

$$= \sum_i \sum_{i \neq j} \vec{F}_{ij} \times \vec{r}_i = \sum_i \sum_{j \neq i} |F_{ij}| \cdot (\vec{r}_i - \vec{r}_j) \times \vec{r}_i$$

$|F_{ij}| = |F_{ji}|$ = 3rd Newton's law

$$= \sum_{pairs} |F_{ij}| \underbrace{(\vec{r}_i - \vec{r}_j) \times \vec{r}_i}_{-\vec{r}_j \times \vec{r}_i} + |F_{ij}| (-1) \underbrace{(\vec{r}_i - \vec{r}_j) \times \vec{r}_j}_{\vec{r}_i \times \vec{r}_j}$$

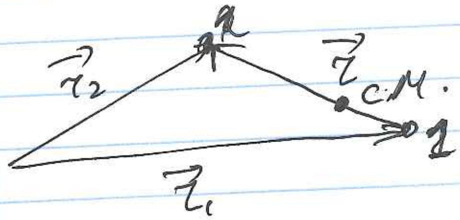
$$\cancel{= \sum_{pairs}}$$

$$= \sum_{pairs} |F_{ij}| \left[\Big(\underbrace{-\vec{r}_j \times \vec{r}_i}_{\substack{\parallel \\ \vec{r}_i \vec{r}_j}} \Big) - (\vec{r}_i \times \vec{r}_j) \right] =$$

$$= \sum_{pairs} |F_{ij}| \cdot 0$$

$$\Rightarrow \frac{d\vec{L}}{dt} = 0 \Rightarrow \boxed{\vec{L} = \text{const}}$$

System of 2 particles



$$\vec r_2 = \vec r_1 + \vec r$$

$$\vec r_{c.M} = \frac{m_1 \vec r_1 + m_2 \vec r_2}{m_1 + m_2}$$

$\Rightarrow \vec r_{c.M} = 0$ C.M. reference frame

$$\frac{m_1 r_1 + m_2 r_2}{m_1 + m_2} = \frac{m_1 \bar r_1 + m_2(\bar r_1 + \bar r)}{m_1 + m_2} = 0$$

$$\boxed{r_1 = -\frac{m_2 \vec r}{m_1 + m_2} = -\frac{\mu}{m_1}\vec r}$$

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ reduced mass

$$\boxed{r_2 = \frac{\mu}{m_2}\vec r}$$

$$\vec L = m_1 \vec v_1 \times \vec r_1 + m_2 \vec v_2 \times \vec r_2 =$$

$$\cancel{= \frac{\mu}{m_1} v_1 \times \vec r + \mu \vec v_2 \times \vec r}$$

$$= m_1 \vec v_1 \times \left(-\frac{\vec r}{m_1}\mu\right) + m_2 \vec v_2 \times \left(\frac{\vec r}{m_2}\mu\right)$$

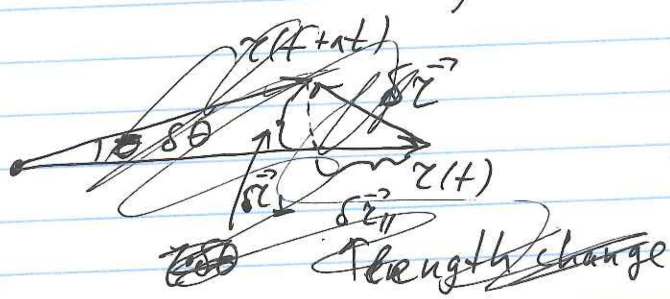
$$\vec v = \dot{\vec r}$$

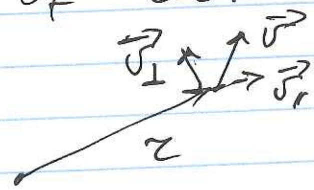
$$= \mu\left(\left(-\vec v_1 + \vec v_2\right) \times \vec r\right.$$

$$= \mu\left(+\frac{\mu}{m_1}\dot{\vec r} + \frac{\mu}{m_2}\dot{\vec r}\right) \times \vec r = \mu \dot{\vec r} \times \vec r$$

$$= \mu\left(+\frac{m_2}{m_1+m_2} + \frac{m_1}{m_1+m_2}\right)\dot{\vec r} \times \vec r = \mu \vec v \times \vec r$$

$\vec{L} = \mu \vec{v} \times \vec{r}$, let's move to polar system of coordinates

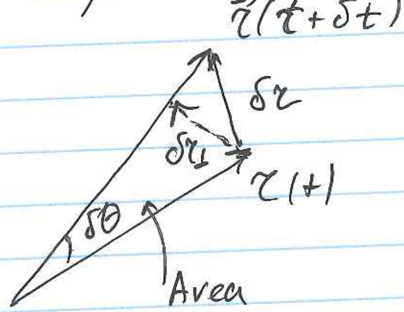




$$\vec{v} = \frac{\delta \vec{r}}{\delta t} =$$

$$\vec{L} = \mu \vec{v} \times \vec{r} = \mu(\vec{v}_{\parallel} + \vec{v}_{\perp}) \times \vec{r} =$$

$$= \mu \vec{v}_{\perp} \times \vec{r} = \mu (v_{\perp} \cdot r) \hat{\theta}$$



$$v_{\perp} = \frac{\delta r_{\perp}}{\delta t} = \frac{r \cdot \delta\theta}{\delta t} =$$

$$= r \cdot \dot{\theta}$$

$$\frac{r \cdot (r\delta\theta)}{2} = \delta \text{Area}$$

$$\Downarrow$$

$$\frac{d\,\text{Area}}{dt} = \frac{1}{2} r^2 \dot{\theta}$$

$$\boxed{|\vec{L}| = \mu r^2 \dot{\theta} = \mu \frac{d\,\text{Area}}{dt} \cdot 2}$$

Kepler's 2nd ~~1st~~ law

Energy conservation

$$\vec{r}_2 - \vec{r}_1$$

$$E = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} + U(r)$$

$$= \frac{m_1 \left(-\frac{\mu}{m_1} v\right)^2}{2} + \frac{m_2}{2}\left(\frac{\mu v}{m_2}\right)^2 + U(r) =$$

$$= \frac{\mu^2}{2} \underbrace{\left(\frac{1}{m_1} + \frac{1}{m_2}\right)}_{\mu^{-1}} v^2 + U(r)$$

$$\boxed{E = \frac{\mu}{2} v^2 + U(r)}$$

$$\boxed{L = \mu \vec{v} \times \vec{r}}$$

$$E = \frac{\mu}{2} v^2 + U(r)$$

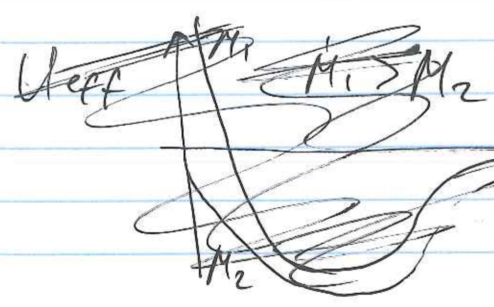
$$\left| \begin{aligned} v^2 &= |\dot{r}\cdot\dot{r}| = \\ &= (\vec{v}_\parallel + \vec{v}_\perp)\cdot(\vec{v}_\parallel + \vec{v}_\perp) \\ &= \underbrace{v_\parallel^2}_{\uparrow\text{change of length}} + v_\perp^2 = (\dot{r})^2 + (r\dot{\theta})^2 \end{aligned} \right|$$

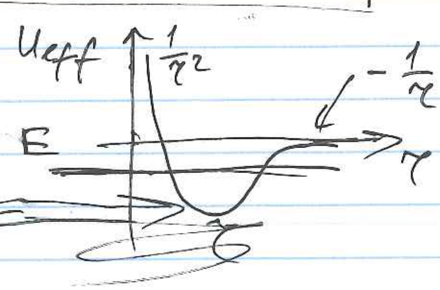
$$E = \frac{\mu(\dot{r})^2}{2} + \frac{\mu(r\dot{\theta})^2}{2} + U(r) =$$

$$= \left| \dot{\theta} = \frac{L}{\mu r^2} \right| =$$

$$= \frac{\mu(\dot{r})^2}{2} + \frac{\mu r^2}{2}\cdot\frac{L^2}{\mu^2 r^4} + U(r) \qquad -\frac{\alpha}{r} = G\frac{M_1 M_2}{r}$$

$$\boxed{E = \underbrace{\frac{\mu(\dot{r})^2}{2}}_{\text{Looks like Kinetic}} + \underbrace{\frac{L^2}{2\mu r^2} + U(r)}_{U_{eff}(r)}}$$





$$U_{eff} = -\frac{\alpha}{r} + \frac{\cancel{\mu\alpha} L}{2\mu r^2}$$

$$\dot{r} = \sqrt{\frac{2}{\mu}\left(E - U_{eff}\right)}$$

$$\frac{dr}{\sqrt{\frac{2}{\mu}(E - U_{eff})}} = dt = \frac{\mu r^2}{L} d\theta$$

$$L = \mu r^2 \frac{d\theta}{dt} \Rightarrow dt = \frac{\mu r^2}{L} d\theta$$

$$\int d\theta = \int \frac{L}{\mu r^2 \sqrt{\frac{2}{\mu}(E - U_{eff})}} dr$$

$$= \int \frac{L/r^2}{\sqrt{2\mu(E - U_{eff})}}$$

$$U(r) = -\frac{\alpha}{r} = -\frac{m_1 m_2}{r} G$$

$\Rightarrow$ Skipping boring math

Notice $\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}}$

See Landau p. 50

$$\theta = \arccos \frac{\frac{L}{r} - \frac{\mu\alpha}{L}}{\sqrt{2\mu E + \frac{\mu^2\alpha^2}{L^2}}}$$

$$\cos\theta = \frac{\frac{L}{r} - \frac{\mu\alpha}{L}}{\sqrt{2\mu E + \frac{\mu^2\alpha^2}{L^2}}} = \cancel{\frac{\mu\alpha}{r}} \frac{\frac{L^2}{\mu\alpha}\frac{1}{r} - 1}{\sqrt{1 + \frac{2EL^2}{\mu\alpha^2}}}$$

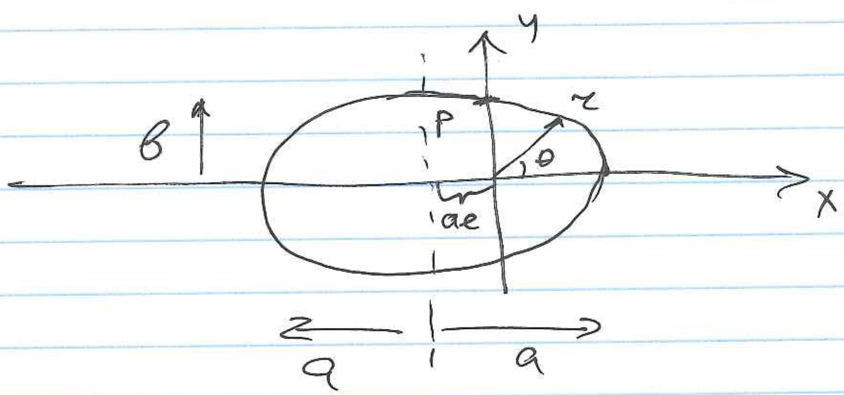
$$\cos\theta = \frac{\frac{L^2}{\mu\alpha}\frac{1}{r} - 1}{\sqrt{1 + \frac{2EL^2}{\mu\alpha^2}}}$$

$$\boxed{e = \sqrt{1 + \frac{2EL^2}{\mu\alpha^2}}}$$

$$\boxed{p = \frac{L^2}{\mu\alpha}}$$

$$\Rightarrow \boxed{\frac{p}{r} = 1 + e\cos\theta}$$

lecture 5 stopped here



$$2a = \frac{p}{1+e} + \frac{p}{1-e} = \frac{p(1-e+1+e)}{1-e^2}$$

$$\Rightarrow \boxed{a = \frac{p}{1-e^2} = \frac{\alpha}{2|E|}}$$

$$\boxed{b = \frac{p}{\sqrt{1-e^2}} = \frac{L}{\sqrt{2\mu|E|}}}$$

$$b = y_{max} = \max\{r \cdot \sin\theta\} = \max\left\{\frac{p}{1+e\cos\theta}\sin\theta\right\}$$