

Root finding

Eugeniy E. Mikhailov

The College of William & Mary



Lecture 05

Notes

Root finding problem

Generally we want to solve the following canonical problem

$$f(x) = 0$$

Notes

Root finding problem

Generally we want to solve the following canonical problem

$$f(x) = 0$$

Example

$$2 \sin(x) - 1 = 0$$

Notes

Root finding problem

Generally we want to solve the following canonical problem

$$f(x) = 0$$

Example

$$2 \sin(x) - 1 = 0$$

Often we have a problem which looks slightly different

$$h(x) = g(x)$$

Notes

Root finding problem

Generally we want to solve the following canonical problem

$$f(x) = 0$$

Example

$$2 \sin(x) - 1 = 0$$

Often we have a problem which looks slightly different

$$h(x) = g(x)$$

But it is easy to transform to canonical form with

$$f(x) = h(x) - g(x) = 0$$

Notes

Root finding problem

Generally we want to solve the following canonical problem

$$f(x) = 0$$

Example

$$2 \sin(x) - 1 = 0$$

Often we have a problem which looks slightly different

$$h(x) = g(x)$$

But it is easy to transform to canonical form with

$$f(x) = h(x) - g(x) = 0$$

Example

$$3x^3 + 2 = \sin x \rightarrow 3x^3 + 2 - \sin x = 0$$

Notes

Trial and error method

One can try to get the solution by just guessing with a hope to hit the solution. This is not highly scientific.

Notes

Trial and error method

One can try to get the solution by just guessing with a hope to hit the solution. This is not highly scientific. However each guess can provide some clues.

Notes

Trial and error method

One can try to get the solution by just guessing with a hope to hit the solution. This is not highly scientific.

However each guess can provide some clues.

- A general search algorithm is the following
- make a guess i.e. trial
 - make intelligent new guess (x_{i+1}) judging from this trial (x_i)
 - continue as long as $|f(x_{i+1})| > \varepsilon_f$ and $|x_{i+1} - x_i| > \varepsilon_x$

Notes

Trial and error method

One can try to get the solution by just guessing with a hope to hit the solution. This is not highly scientific.

However each guess can provide some clues.

- A general search algorithm is the following
- make a guess i.e. trial
 - make intelligent new guess (x_{i+1}) judging from this trial (x_i)
 - continue as long as $|f(x_{i+1})| > \varepsilon_f$ and $|x_{i+1} - x_i| > \varepsilon_x$

- Example
- Let's play a simple game:
- some one think of any number between 1 and 100
 - I will make a guess
 - you provide me with either "less" or "more" depending where is my guess with respect to your number
- How many guesses do I need?

Notes

Trial and error method

One can try to get the solution by just guessing with a hope to hit the solution. This is not highly scientific.

However each guess can provide some clues.

- A general search algorithm is the following
- make a guess i.e. trial
 - make intelligent new guess (x_{i+1}) judging from this trial (x_i)
 - continue as long as $|f(x_{i+1})| > \varepsilon_f$ and $|x_{i+1} - x_i| > \varepsilon_x$

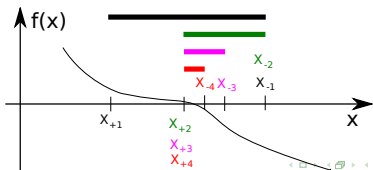
- Example
- Let's play a simple game:
- some one think of any number between 1 and 100
 - I will make a guess
 - you provide me with either "less" or "more" depending where is my guess with respect to your number
- How many guesses do I need? At most 7

Notes

Bisection method pseudo code

Works for any **continuous function** in vicinity of function root

- make initial bracket for search x_+ and x_- such that
 - $f(x_+) > 0$
 - $f(x_-) < 0$
- loop begins
- make new guess value $x_g = (x_+ + x_-)/2$
- if $|f(x_g)| \leq \varepsilon_f$ or $|x_+ - x_g| \leq \varepsilon_x$
stop we found the solution with desired approximation
- otherwise if $f(x_g) > 0$ then $x_+ = x_g$ else $x_- = x_g$
- continue the loop



Notes

Bisection - simplified matlab implementation

```
function x_sol=bisection(f, xn, xp, eps_f, eps_x)
% solving f(x)=0 with bisection method

xg=(xp+xn)/2; % initial guess
fg=f(xg); % initial function evaluation

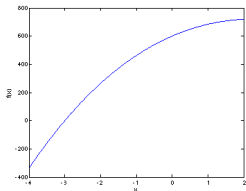
while ( (abs(fg) > eps_f) & (abs(xg-xp)>eps_x) )
    if (fg>0)
        xp=xg;
    else
        xn=xg;
    end
    xg=(xp+xn)/2; % update guess
    fg=f(xg); % update function evaluation
end
x_sol=xg; % solution is ready
end
```

Notes

Bisection - example of use

Let's define simple test function in the file 'function_to_solve.m'

```
function ret=function_to_solve(x)
ret=(x-10)*(x-20)*(x+3);
end
```

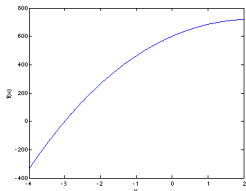


Notes

Bisection - example of use

Let's define simple test function in the file 'function_to_solve.m'

```
function ret=function_to_solve(x)
ret=(x-10)*(x-20)*(x+3);
end
```



pay attention to the function handle operator @

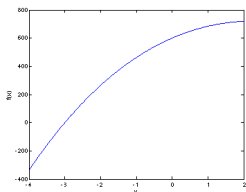
```
eps_x=1e-8;
eps_f=1e-6;
x0=bisection(...
    @function_to_solve,...
    -4.1, 2, ...
    eps_f, eps_x)
```

Notes

Bisection - example of use

Let's define simple test function in the file 'function_to_solve.m'

```
function ret=function_to_solve(x)
ret=(x-10)*(x-20)*(x+3);
end
```



pay attention to the function handle operator @

```
eps_x=1e-8;
eps_f=1e-6;
x0=bisection(...
    @function_to_solve,...
    -4.1, 2, ...
    eps_f, eps_x)
```

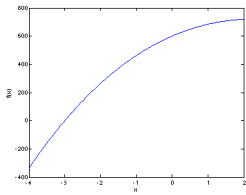
x0 = -3.0000

Notes

Bisection - example of use

Let's define simple test function in the file 'function_to_solve.m'

```
function ret=function_to_solve(x)
    ret=(x-10)*(x-20)*(x+3);
end
```



pay attention to the function handle operator @

```
eps_x=1e-8;
eps_f=1e-6;
x0=bisection(...
    @function_to_solve,...
    -4.1, 2, ...
    eps_f, eps_x)
```

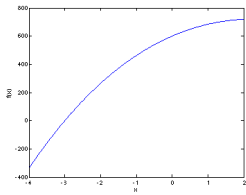
always cross check results

```
x0 = -3.0000
>> function_to_solve(x0)
ans = 3.0631e-07
```

Bisection - example of use

Let's define simple test function in the file 'function_to_solve.m'

```
function ret=function_to_solve(x)
    ret=(x-10)*(x-20)*(x+3);
end
```



pay attention to the function handle operator @

```
eps_x=1e-8;
eps_f=1e-6;
x0=bisection(...
    @function_to_solve,...
    -4.1, 2, ...
    eps_f, eps_x)
```

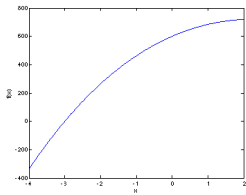
always cross check results

```
x0 = -3.0000
>> function_to_solve(x0)
ans = 3.0631e-07
```

Bisection - example of use

Let's define simple test function in the file 'function_to_solve.m'

```
function ret=function_to_solve(x)
    ret=(x-10)*(x-20)*(x+3);
end
```



pay attention to the function handle operator @

```
eps_x=1e-8;
eps_f=1e-6;
x0=bisection(...
    @function_to_solve,...
    -4.1, 2, ...
    eps_f, eps_x)
```

always cross check results

```
x0 = -3.0000
>> function_to_solve(x0)
ans = 3.0631e-07
```

What is missing in the bisection code?

Notes

Notes

Notes

Notes

What is missing in the bisection code?

The simplified bisection code is missing validation of input arguments.

Notes

What is missing in the bisection code?

The simplified bisection code is missing validation of input arguments.
People make mistakes, typos and all sorts of misuse.

Notes

What is missing in the bisection code?

The simplified bisection code is missing validation of input arguments.
People make mistakes, typos and all sorts of misuse.

"If something can go wrong it will"

Muphry's Law

Notes

What is missing in the bisection code?

The simplified bisection code is missing validation of input arguments.
People make mistakes, typos and all sorts of misuse.

"If something can go wrong it will"

Muphry's Law

Never expect that user will put valid inputs.

Notes

What is missing in the bisection code?

The simplified bisection code is missing validation of input arguments. People make mistakes, typos and all sorts of misuse.

"If something can go wrong it will"

Muphry's Law

Never expect that user will put valid inputs.

So what should we check for sure

- 1 $f(x_n) < 0$
- 2 $f(x_p) > 0$

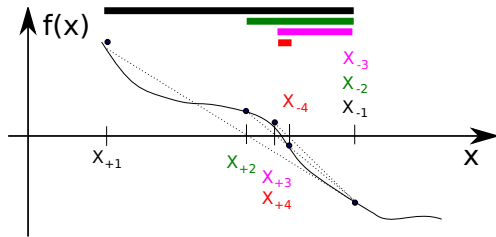
It would be handy to return secondary outputs

- with the value of function at the found solution point
- the number of iterations (good for performance tests)

Eugeniy Mikhailov (W&M) Practical Computing Lecture 05 7 / 10

False position (*regula falsi*) method

In this method we naively approximate our function as a line.



Eugeniy Mikhailov (W&M) Practical Computing Lecture 05 8 / 10

False position method - pseudo code

- make initial bracket for search x_+ and x_- such that
 - $f(x_+) > 0$
 - $f(x_-) < 0$
- loop begins
- draw a chord between points $(x_-, f(x_-))$ and $(x_+, f(x_+))$
- make new guess value at the point of the chord intersection with the 'x' axis

$$x_g = \frac{x_- f(x_+) - x_+ f(x_-)}{f(x_+) - f(x_-)}$$
- if $|f(x_g)| \leq \epsilon_f$ or $|x_+ - x_g| \leq \epsilon_x$ stop we found the solution with desired approximation
- otherwise if $f(x_g) > 0$ then $x_+ = x_g$ else $x_- = x_g$
- continue the loop

Note: it looks like bisection except the way of updating x_g

Eugeniy Mikhailov (W&M) Practical Computing Lecture 05 9 / 10

Solution convergence

We say that algorithm has defined convergence if it is possible to express

$$\lim_{k \rightarrow \infty} (x_{k+1} - x_0) = c(x_k - x_0)^m$$

Where x_0 is true root of the equation, c is some constant, and m is the order of convergence.

Notes

Notes

Notes

Notes

Solution convergence

We say that algorithm has defined convergence if it is possible to express

$$\lim_{k \rightarrow \infty} (x_{k+1} - x_0) = c(x_k - x_0)^m$$

Where x_0 is true root of the equation, c is some constant, and m is the order of convergence.

The best algorithm have quadratic convergence i.e. $m = 2$

Notes

Solution convergence

We say that algorithm has defined convergence if it is possible to express

$$\lim_{k \rightarrow \infty} (x_{k+1} - x_0) = c(x_k - x_0)^m$$

Where x_0 is true root of the equation, c is some constant, and m is the order of convergence.

The best algorithm have quadratic convergence i.e. $m = 2$

- the bisection algorithm has linear rate of convergence ($m = 1$) and $c = 1/2$
- it is generally impossible to define convergence order for the false position method

Notes

Solution convergence

We say that algorithm has defined convergence if it is possible to express

$$\lim_{k \rightarrow \infty} (x_{k+1} - x_0) = c(x_k - x_0)^m$$

Where x_0 is true root of the equation, c is some constant, and m is the order of convergence.

The best algorithm have quadratic convergence i.e. $m = 2$

- the bisection algorithm has linear rate of convergence ($m = 1$) and $c = 1/2$
- it is generally impossible to define convergence order for the false position method

Generally the speed of the algorithm is related to its convergence order. However other factors may affect the speed.

Notes

Notes
