Digital filters

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Lecture 26

DFT filters (repeat)

Once you get a signal you can filter unwanted components out of it. The recipe is the following

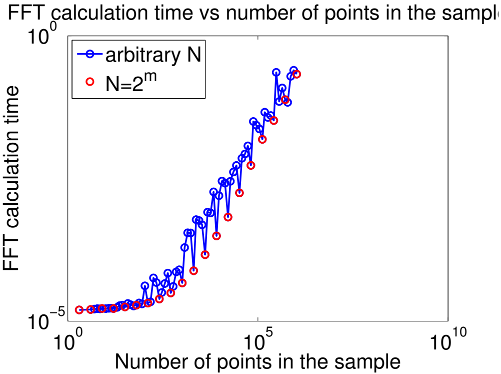
- sample the signal
- calculate forward FT (fft)
- have a look at the spectrum and decide which components are unwanted
- apply filter which attenuate unwanted frequency component (remember that if you attenuate the component of the frequency f by g_f you need to attenuate the component at $-f$ by g_f^*.
- calculate inverse FT (ifft) of the filtered spectrum
- repeat if needed

$$y_{filtered}(t) = \mathcal{F}^{-1}\left[\mathcal{F}(y(t))G(f)\right] = \mathcal{F}^{-1}\left[Y(f)G(f)\right]$$

Speed of FFT

- The main work horse of the DFT filters is FFT algorithm
- it is handy to know its performance behavior

$$y_k = \frac{1}{N}\sum_{n=0}^{N-1} c_n \exp(i\frac{2\pi(k-1)n}{N}) \quad \text{inverse Fourier transform}$$

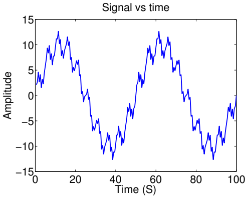


- General DFT scales $\sim N^2$ (N coefficient each involving sum of N)
- FFT scales $\sim N \log_2 N$ which is great speed up

The fastest calculation time for

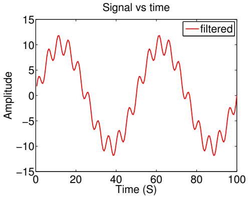
$$N = 2^m$$

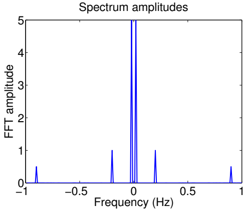
Brick wall low-pass filter

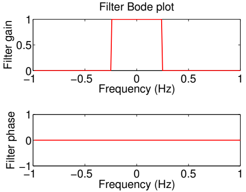


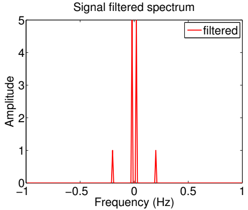
Filter gain function

$$G(f) = \begin{cases} 1, |f| \leq f_{cutoff} \\ 0, |f| > f_{cutoff} \end{cases}$$





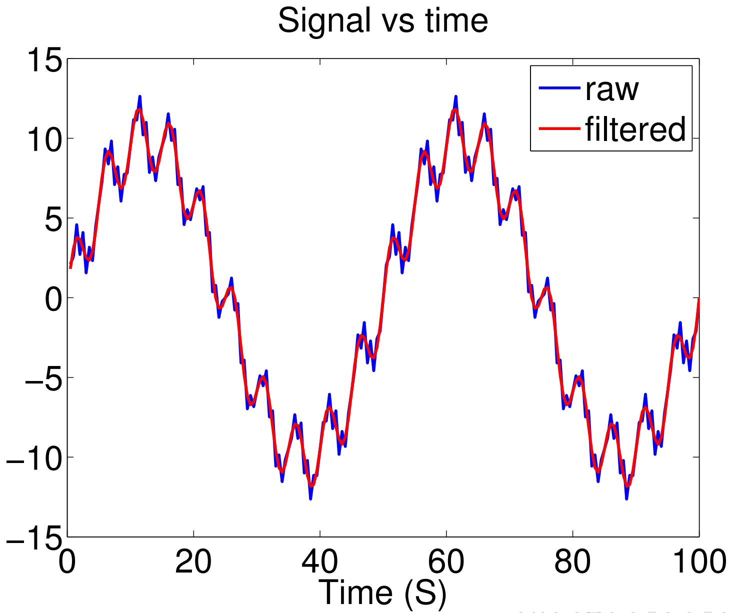




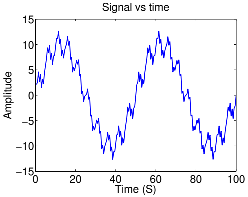
$$y_{filtered}(t) = \mathcal{F}^{-1}\left[\mathcal{F}(y(t))G(f)\right] = \mathcal{F}^{-1}\left[Y(f)G(f)\right]$$

```
freq=fourier_frequencies(SampleRate, N);
G=ones(N,1); G( abs(freq) > Fcutoff, 1)= 0;
y_filtered  =  ifft(  fft( y ) .* G  )
```

Brick wall low-pass filter (continued)

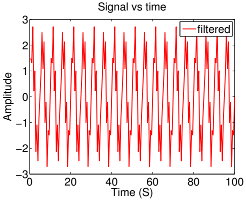


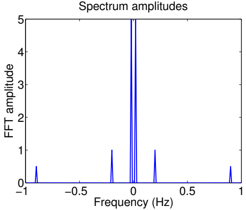
Brick wall high-pass filter

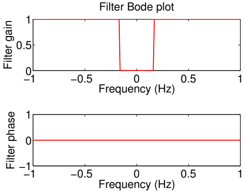


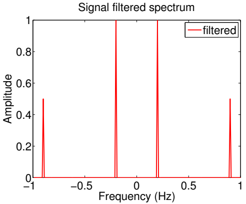
Filter gain function

$$G(f) = \begin{cases} 1, |f| \geq f_{cutoff} \\ 0, |f| < f_{cutoff} \end{cases}$$





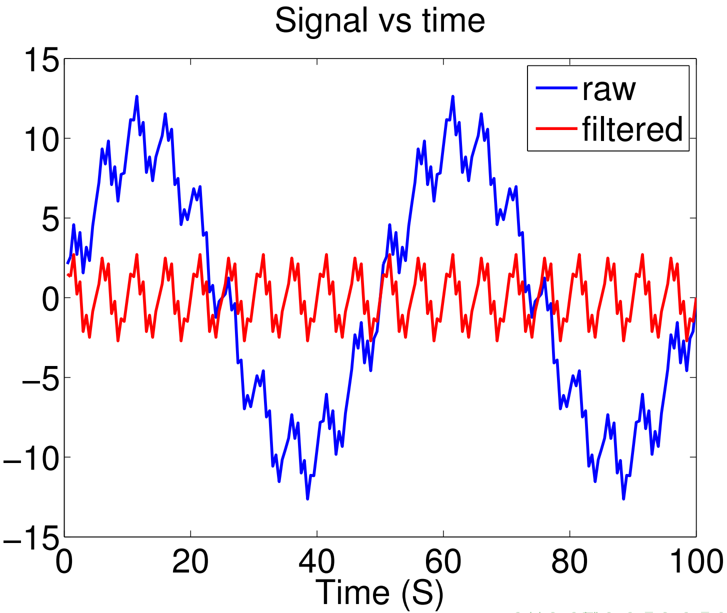




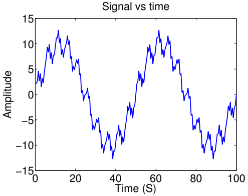
$$y_{filtered}(t) = \mathcal{F}^{-1}\left[\mathcal{F}(y(t))G(f)\right] = \mathcal{F}^{-1}\left[Y(f)G(f)\right]$$

```
freq=fourier_frequencies(SampleRate, N);
G=ones(N,1); G( abs(freq) < Fcutoff, 1)= 0;
y_filtered = ifft( fft( y ) .* G )
```

Brick wall high-pass filter (continued)

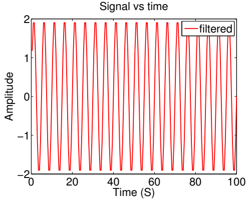


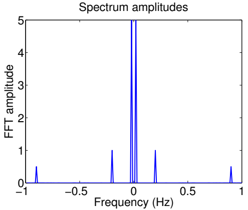
Brick wall band-pass filter

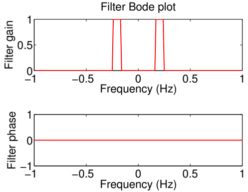


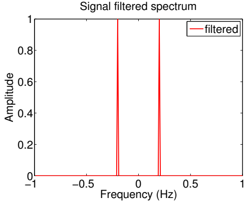
Filter gain function

$$G(f) = \begin{cases} 1, & ||f| - f_c| \le \frac{f_{bw}}{2} \\ 0, & ||f| - f_c| > \frac{f_{bw}}{2} \end{cases}$$



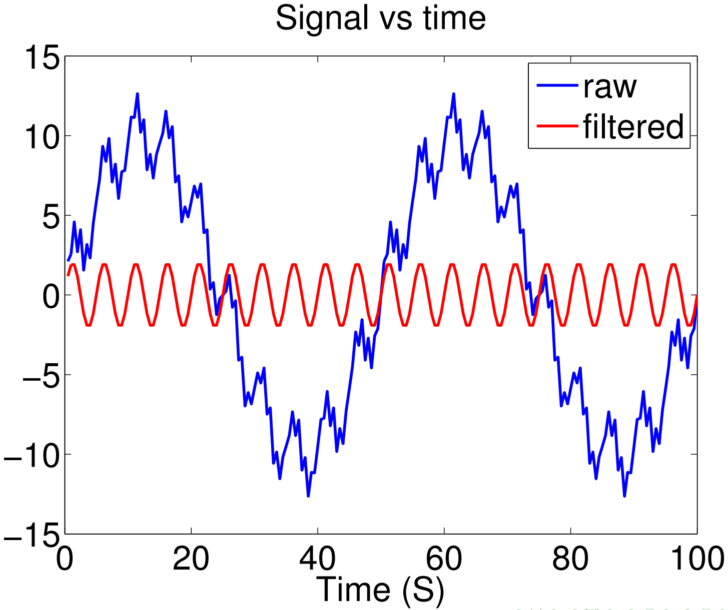




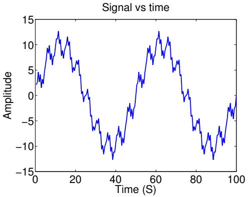


$$y_{filtered}(t) = \mathcal{F}^{-1}\left[\mathcal{F}(y(t))G(f)\right] = \mathcal{F}^{-1}\left[Y(f)G(f)\right]$$

```
freq=fourier_frequencies(SampleRate, N);
G=ones(N,1); G( abs(abs(freq)-Fcenter) > BW/2, 1)=0;
y_filtered  =  ifft(  fft( y ) .* G  )
```

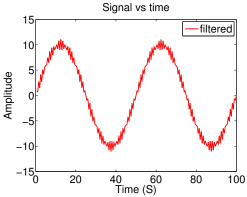

Signal vs time
15
10
5
0
–5
–10
–15
0
20
40
60
80
100
Time (S)
raw
filtered

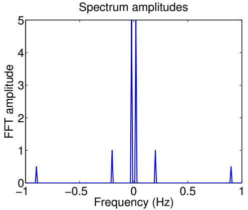
Brick wall band-stop filter

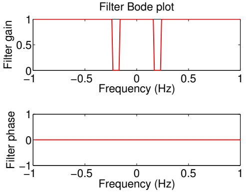


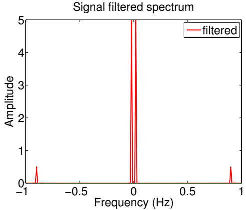
Filter gain function

$$G(f) = \begin{cases} 0, & ||f| - f_c| \leq \frac{f_{bw}}{2} \\ 1, & ||f| - f_c| > \frac{f_{bw}}{2} \end{cases}$$





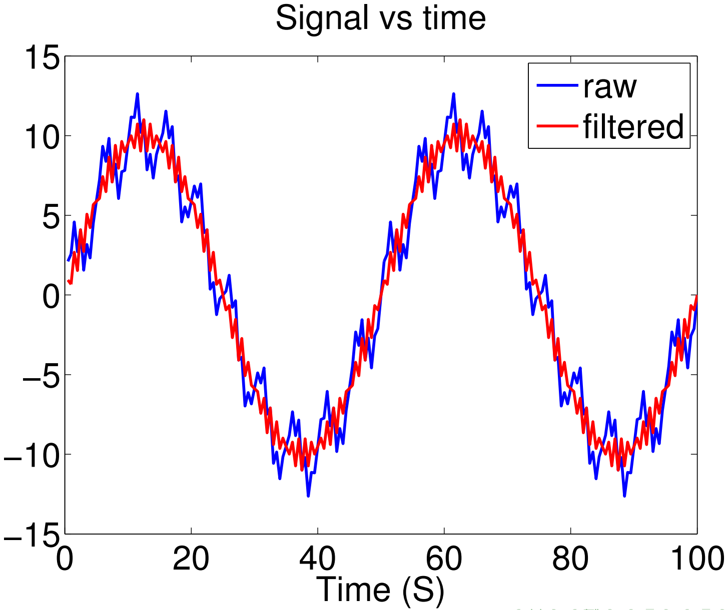




$$y_{filtered}(t) = \mathcal{F}^{-1}\left[\mathcal{F}(y(t))G(f)\right] = \mathcal{F}^{-1}\left[Y(f)G(f)\right]$$

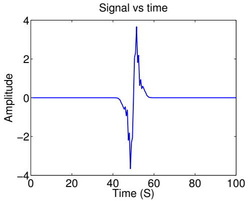
```
freq=fourier_frequencies(SampleRate, N);
G=zeros(N,1); G( abs(abs(freq)-Fcenter) > BW/2, 1)=1;
y_filtered  =  ifft(  fft( y ) .* G  )
```

Brick wall band-stop filter (continued)



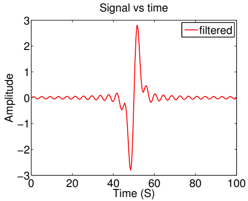
Brick wall filters artifacts

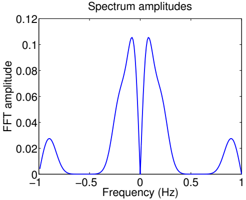
Sharp features in Fourier spectrum produce ring-down like signals

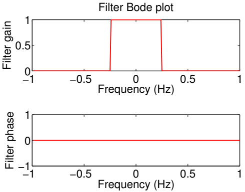


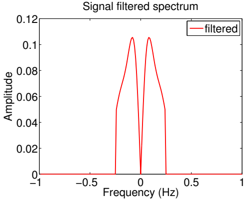
Filter gain function

$$G(f) = \begin{cases} 1, |f| \leq f_{cutoff} \\ 0, |f| > f_{cutoff} \end{cases}$$



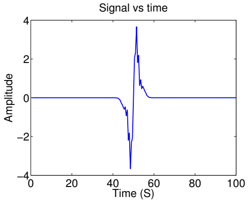


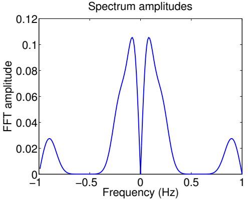




Low pass smoothed

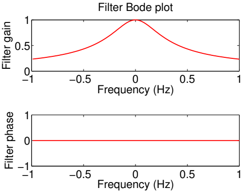
Sharp features in Fourier spectrum produce ring-down like signals

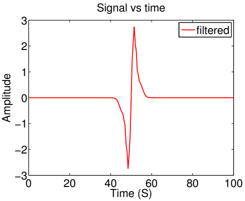


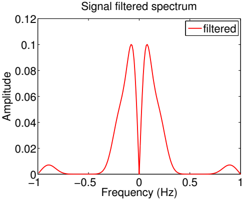


Filter gain function

$$G(f) = \left| \frac{1}{1 + if/f_{cutoff}} \right|$$

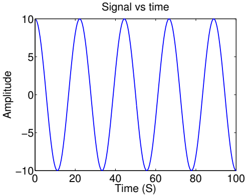






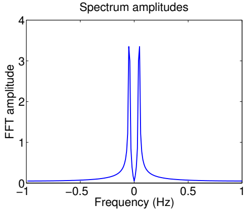
Windowing artifacts

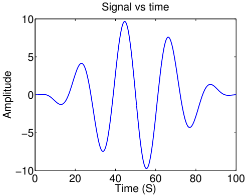
Similarly sharp features in time lead to broadening of the spectrum



Implicitly assumed
Rectangular window

$$w_n = 1$$

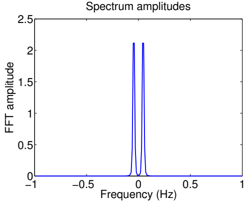




$$y_{windowed_n} = y_n w_n$$

Hann window

$$w_n = \frac{1}{2}\left(1 - \cos\left(2\pi\frac{n-1}{N-1}\right)\right)$$



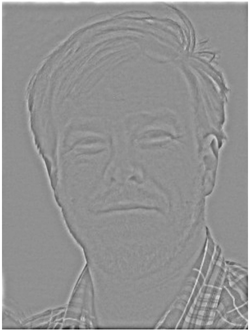
- Note: spectral resolution $\sim 1/T_{window}$.

Search for other windowing functions: Hamming, Tukey, Cosine, Lanczos, Triangulars, Gaussians, Bartlett-Hann, Blackmans, Kaisers. They all drop a signal at the beginning and at the end to zero.

Other DFT applications

Fun one: two dimensional high and low pass image filter with merge





Depending on distance to the image you should see either me or Prof. Novikova in the middle.
To see the second image either step aside or decrease zoom till you do not see details on the right most image.
If you can take of your glasses the illusion is stronger.