Ordinary Differential equations

Eugeniy E. Mikhailov

The College of William & Mary

Lecture 19

ODE definitions

An ordinary equation of order n has the following form

$$y^{(n)} = f(x, y, y', y'', \cdots, y^{(n-1)})$$

- x independent variable
- $y^{(i)}$ the i_{th} derivative of $y(x)$
- f often called the force term

First order ODE example

Example

the acceleration of the body is the first derivative of velocity with respect to the time and equals to the force divided by mass

$$a(t) = \frac{dv}{dt} = v'(t) = \frac{F}{m}$$

$$\begin{aligned} t &\to x \text{ independent variable} \\ v &\to y \\ F/m &\to f \end{aligned}$$

And we obtain the canonical form

$$y^{(1)} = f(x, y)$$

for the first order ODE

n_{th} order ODE transformation to the system of first order ODE

$$y^{(n)} = f(x, y, y', y'', \cdots, y^{(n-1)})$$

we define the following variables

$$y_1 = y, y_2 = y', y_3 = y'', \cdots, y_n = y^{(n-1)}$$

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \\ \vdots \\ y_{n-1}' \\ y_n' \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{n-1} \\ f_n \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ y_4 \\ \vdots \\ y_n \\ f(x, y_1, y_2, y_3, \cdots y_n) \end{pmatrix}$$

So we can rewrite n_{th} order ODE as a system of first order ODE

$$\vec{y}' = \vec{f}(x, \vec{y})$$

Cauchy boundary conditions

$$\vec{y}' = \vec{f}(x, \vec{y})$$

This is the system of n equations and thus requires n constrains.

With Cauchy boundary conditions we specify $\vec{y}(x_0) = \vec{y}_0$ i.e. initial conditions

$$\begin{pmatrix} y_1(x_0) \\ y_2(x_0) \\ y_3(x_0) \\ \vdots \\ y_n(x_0) \end{pmatrix} = \begin{pmatrix} y_{1_0} \\ y_{2_0} \\ y_{3_0} \\ \vdots \\ y_{n_0} \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \\ y_0'' \\ \vdots \\ y_0^{(n-1)} \end{pmatrix}$$

Problem example

If acceleration of the particle is given and constant find the position as a function of time.
We are solving

$$x''(t) = a$$

First we need to convert it to canonical form of system of first order ODEs.

$$\begin{aligned} t &\rightarrow x \text{ time as independent variable} \\ x &\rightarrow y \rightarrow y_1 \text{ particle position} \\ v &\rightarrow y' \rightarrow y_2 \text{ velocity} \\ a &\rightarrow f \text{ acceleration as a force term} \end{aligned}$$

so

$$x'' = a \rightarrow y'' = f \rightarrow \vec{y}' = \vec{f}(x, \vec{y}) \rightarrow \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2 \\ f \end{pmatrix}$$

We also need initial conditions:
initial position $x_0 \rightarrow y_{1_0}$ and velocity $v_0 \rightarrow y_{2_0}$

Euler's method

Let's for simplicity consider simple first order ODE (notice lack of vector)

$$y' = f(x, y)$$

There is an exact way to write the soltion

$$y(x_f) = \int_{x_0}^{x_f} f(x, y) dx$$

The problem is that $f(x, y)$ depends on y itself. However for small interval of $x, x + h$ we can assume that $f(x, y)$ is constant
Then we get familiar box integration formula or in application to ODE the Euler's method.

$$y(x + h) - y(h) = \int_{x}^{x+h} f(x, y) dx \approx f(x, h)h$$

Euler's method continued

$$y(x+h) = y(x) + f(x,y)h$$

All we need is to split our interval on bunch of steps of size h, and leap frog from the first x_0 to the next one $x_0 + h$, then $x_0 + 2h$ and so on. Now we can make an easy transformation to the vector case (i.e. n_{th} order ODE)

$$\vec{y}(x+h) = \vec{y}(x) + \vec{f}(x,y)h$$

Note: similarly to the boxes integration method
Euler's method is very imprecise for the given h

Stability issue

Let's have a look at the first oder ODE

$$y' = 3y - 4e^{-x}$$

It has the following solution

$$y = Ce^{3x} + e^{-x}$$

If our initial condition $y(0) = 1$ the solution is $y(x) = e^{-x}$.
Please run *ode_unstable_example.m* and have a look at the output of the numerical solution
Clearly it's diverges from the analytical solutions. The problem is in round off errors which is the same as to say that $y(0) = 1 + \delta$ then $C \neq 0$ and solution diverges.
Do nut trust the numerical solutions (regardless of the method) without proper consideration!