Combinatorial optimization

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Lecture 17

Combinatorial optimization problem statement

We still want to optimize (minimize) our multi dimensional merit function E

Find $\vec{x}$ that minimize $E(\vec{x})$ subject to $g(\vec{x}) = 0, h(\vec{x}) \leq 0$

The only difference values of $\vec{x}$ are discrete. I.e. any component of $\vec{x}$ can take a **countable** set of different values.

In this case we cannot run our golden search algorithm or anything else which assumes continuous space for $\vec{x}$.

Instead we have to find a method to search through discrete sets of all possible input values, i.e. go through all possible combinations of $\vec{x}$ components.

Hence, the name combinatorial optimization.

Example: Backpack problem

Suppose you have a backpack with a given size (volume). You have a set of objects with given volumes and values (for example their cost).

Our job is to find a such subset of items that still fits in the backpack and has the maximum combined value.

In simple case we will assume that every items happen only once. Then our job is to maximize

$$E(\vec{x}) = \sum \text{value}_i x_i = \overrightarrow{\text{values}} \cdot \vec{x}$$

Subject of the following constrains

$$\sum \text{volume}_i x_i = \overrightarrow{\text{volumes}} \cdot \vec{x} \leq \text{BackpackSize}$$

Where $x_i = (0 \text{ or } 1)$ i.e. do we take this object or not

Brute force optimization

With this approach we will just try all possible combinations of items and find the best of them.

Notice that if there is N objects we have 2^N of all possible combinations to choose from.

So the size of the problem space and thus probing time grows very fast.

Backpack optimization: new test set generation

Recall that we are looking for an optimal direction among all possible $\vec{x}$

Generally $\vec{x}$ is combo of zeros and ones $\vec{x} = [0, 1, 0, 1, \cdots, 1, 1, 0, 1, 1]$

How would we generate all possible combinations of $\vec{x}$ components?

Backpack optimization: new test set generation

Recall that we are looking for an optimal direction among all possible $\vec{x}$
Generally $\vec{x}$ is combo of zeros and ones $\vec{x} = [0, 1, 0, 1, \cdots, 1, 1, 0, 1, 1]$
How would we generate all possible combinations of $\vec{x}$ components?

- $\vec{x}$ looks like binary number.
- let's start with $\vec{x} = [0, 0, 0, 0, \cdots, 0, 0]$
- every new components will be generated by adding 1 to the previous x according to binary addition rules
  - for example
    $x_{next} = [1, 0, 1, \cdots, 1, 1, 0, 1, 1] + 1 = [1, 0, 1, \cdots, 1, 1, 1, 0, 0]$
- for every new $\vec{x}$ we check if items fit to backpack and if new fitted value is larger then previous
- once we tried all 2^N combinations of $\vec{x}$ we are done

- So time of optimization grows exponentially with the number N of items to chose.
- But we will find the global optimum.

Backpack optimization: test run

For realization of this algorithm have a look at the
backpack_binary.m
Sample run

```
backpack_size=7;
volumes=[  2,  5,  1,  3, 3];
values =[ 10, 12, 23, 45, 4];
[pbest, max_fitted_value] = ...
  backpack_binary( backpack_size, volumes, values)

pbest = [1 3 4]
max_fitted_value = 78
```

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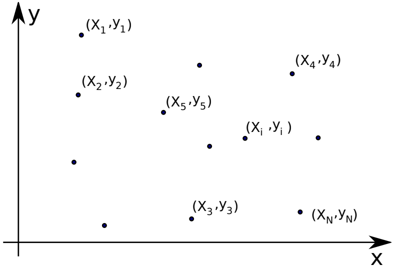
- My computer sorts 20 items in 47 seconds, but 30 items would take 1000 times longer something like 13 hours to solve.

Example: Traveling salesman problem

This problem has a lot of connection to the real life. Every time you ask your GPS to find a route, the GPS unit has to solve this problem. Layout of traces on a printed circuits board is essentially the same problem as well.

- Suppose you have N cities (with given coordinates) to visit
- Salesman start in the city 1 and need to be in the city *N* at the end of the route
- Find the shortest route so salesman visits every city only once



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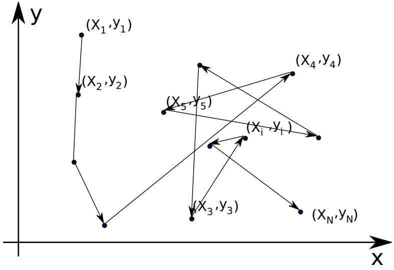

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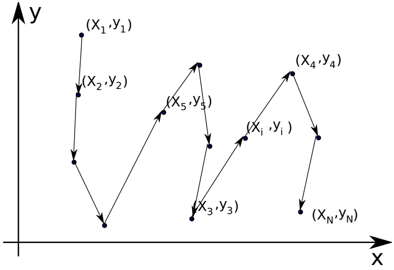

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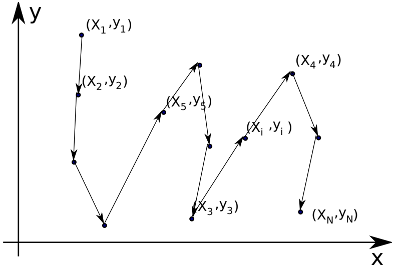

Note that combinatorial complexity of this problem

$$(N-2)!$$

since ends points are fixed. This grows very fast with N.

Possible solutions

- Brute force - combinatorial
  - Try every possible combination of cities and choose the best one
  - Will work for the modest route with $N \leq 10$ or may be slightly more

Permutation generating algorithm

The below method goes back to 14th century India. It generates permutations in the lexicographical order.

1. find the largest index k such that $p(k) < p(k+1)$.
   - if no such index exists, the permutation is the last permutation.
2. find the largest index l such that $p(k) < p(l)$.
   - There is at least one $l = k + 1$
3. swap $a(k)$ with $a(l)$.
4. reverse the sequence from $a(k+1)$ up to and including the final element $a(end)$.

See the complimentary code '`permutation.m`'