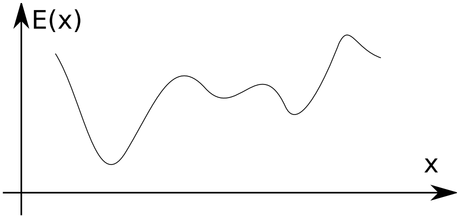
Optimization problem

Eugeniy E. Mikhailov

The College of William & Mary

Lecture 13

Introduction to optimization



Find $\vec{x}$ that minimize $E(\vec{x})$ subject to $g(\vec{x}) = 0, h(\vec{x}) \leq 0$

$\vec{x}$ design variables

$E(\vec{x})$ merit or objective or fitness or energy function

$g(\vec{x})$ and $h(\vec{x})$ constrains

There is no guaranteed way (algorithm) which can find global minimum (optimal) point.

Easy to see that maximization problem is the same as minimization once $E(\vec{x}) \to -E(\vec{x})$.

Analytical solution of 1D

If we have 1D case and $E(x)$ has analytical derivative, optimization problem can be restated as

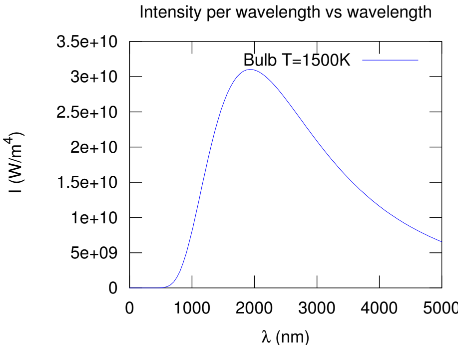
Find $f(x) = 0$
where $f(x) = dE/dx$

since at maximum or minimum derivative must be zero.
Since we already know how to find the solution of $f(x) = 0$ the rest is easy.

Example: max of black body radiation spectrum

According to Plank's law energy density per of black body radiation

$$I(\lambda, T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda k T}} - 1}$$



where

- h is Planck constant 6.626×10^{-34} J×s,
- c is speed of light 2.998×10^{8} m/s,
- k is Boltzmann constant 1.380×10^{-23} J K^{-1},
- T is body temperature,
- λ is wavelength

Solution with Matlab built in 1D minimization - fminbnd

```
function I_lambda=black_body_radiation(lambda,T)
% black body radiation spectrum
% lambda - wavelength of EM wave
% T - temperature of a black body
h=6.626e-34;  % Plank constant
c=2.998e8;    % speed of light
k=1.380e-23;  % Boltzmann constant

I_lambda = 2*h*c^2 ./ (lambda.^5)  ./ (exp(h*c./(lambda*k*T))-1);
end
```

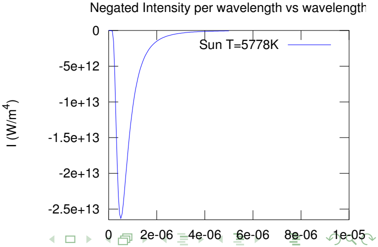
First we flip/negate function since our algorithm is suited form min search and set particular T

```
T=5778;
f = @(x) - black_body_radiation(x,T);
```

Next, we find optimal solution

```
fminbnd(f,1e-9,2e-6,optimset('TolX',1e-12))
ans = 5.0176e-07
% i.e. maximum radiation is at 502 nm
```

Then we plot it to find a bracket



Golden section search algorithm

If you have an initial bracket for solution i.e. found a, b points such that there is a point x_p satisfying $a < x_p < b$ and $E(x_p) < min(E(a), E(b))$. Then $h = (b - a)$

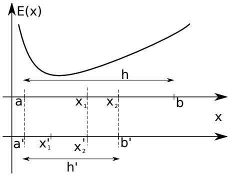
1. assign new probe points $x_1 = a + R * h$ and $x_2 = b - R * h$
2. $E_1 = E(x_1)$, $E_2 = E(x_2)$, $E_a = E(a)$, $E_b = E(b)$
3. if $h < \varepsilon_x$ then stop otherwise do steps below
4. note that for small enough h: $E(x_1) < E(a)$ and $E(x2) < E(b)$
5. shrink/update the bracket
   - if $E_1 < E_2$ then $b = x_2$, $E_b = E_2$ else $a = x_1$, $E_a = E_1$
6. update $h = (b - a)$ and assign new probe points, with the proper R we can reuse one of the old points either x_1, E_1 or x_2, E_2
   - if $E_1 < E_2$
     then $x_2 = x_1$, $E_2 = E_1$, $x_1 = a + R * h$, $E_1 = E(x_1)$
     else $x_1 = x_2$, $E_1 = E_2$, $x_2 = b - R * h$, $E_2 = E(x_2)$
7. go to step 3

R given by the golden section $R = \frac{3-\sqrt{5}}{2} \approx 0.38197$

Derivation of the R value

at first step we have

$$
\begin{aligned}
x_1 &= a + R * h \\
x_2 &= b - R * h
\end{aligned}
$$



Suppose that $E(x_1) < E(x_2)$ then $a' = a$ and $b' = x_2$ then for the next bracket we evaluate x_1' and x_2'

$$
\begin{aligned}
x_1' &= a' + R * h' = a' + R * (b' - a') \\
x_2' &= b' - R * h' = b' - R * (b' - a') \\
&= x_2 - R * (x_2 - a) = b - R * h - R * (b - R * h - a)
\end{aligned}
$$

we would like to reuse on of the previous evaluations of E so we require that $x_1 = x_2'$. This leads to equation

$$R^2 - 3R + 1 = 0 \text{ with } R = \frac{3 \pm \sqrt{5}}{2}$$

We need to choose minus sign since fraction $R < 1$