Numerical integration continued

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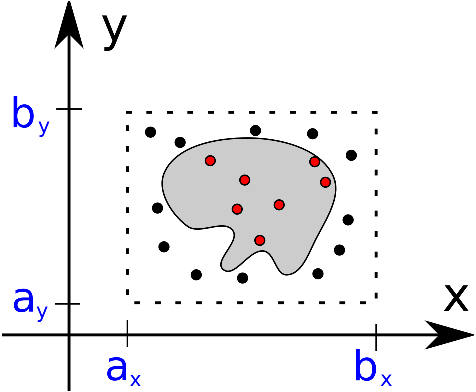
Lecture 10

Midterm project

Problem (100 points total)
You are working for NASA. Your team is responsible to design a rocket which should lift off and after travel time $T_t = 50$ seconds in the gravity field of the Earth must reach a certain orbit with the final verical velocity $v_f = 0$. Do not worry about horizontal velocity, it is another team responsibility.
Engineers provided you with an engine capable to provide to the rocket a time dependent lift acceleration in the form of $a(t) = 100 * \exp(-(\tanh(b * t) * b * t/10)^2)$ (when other forces are disregarded) during time till a fuel line is cut off $T_c = 10$ seconds. The acceleration grows with time since rocket burns fuel and becomes lighter. However at time T_c no fuel is left and thus no lift force provided.
Assume that rocket starts from the planet Earth, treat the acceleration due to gravity as a constant $g = 9.8$ m/s^2 (i.e. neglect gravitational force change). Disregard the air drag.
Task 1 (60 points): Your job is to **find the proper value of coefficient** b. Do not forget the units.
Task 2 (40 points): Plot velocity of the rocket as a function of time once the proper value of b is found.

Bonus is harder but it is within a reach!
Bonus (10 points): Plot the altitude of the rocket as a function of time. What is the altitude of the rocket at time T_t?

Toy example - area of the pond



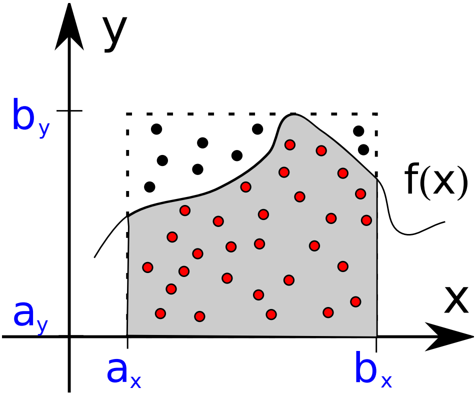
$$A_{pond} = \frac{N_{inside}}{N_{total}} A_{box}$$

where

$$A_{box} = (b_x - a_x)(b_y - a_y)$$

- Points must be uniformly and randomly distributed across the area.
- The smaller the enclosing box the better it is.

Naive Monte Carlo integration



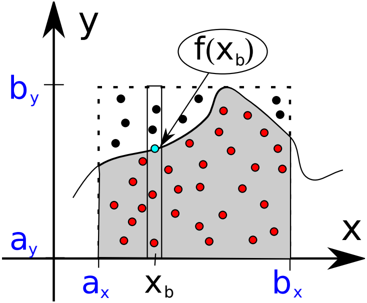
$$\int_{a_x}^{b_x} f(x)dx = \frac{N_{inside}}{N_{total}} A_{box}$$

where

$$A_{box} = (b_x - a_x)(b_y)$$

- Points must be uniformly and randomly distributed across the area.
- The smaller the enclosing box the better it is. So $max(f(x)) \rightarrow b_y$

Monte Carlo integration derived



Notice that if we choose a small stripe around the bin value x_b, then subset of points in that stripe gives an estimate for $f(x_b)$.
Thus why bother spreading points around area?

Let's chose a uniform random distribution of points x_i inside $[a_x, b_x]$

$$\int_{a_x}^{b_x} f(x)dx \approx \frac{b_x - a_x}{N} \sum_{i=1}^{N} f(x_i)$$

Error estimate for Monte-Carlo method

It can be shown that error of the numerical integration (E) is given by the following expressions

Monte Carlo method

$$E = \mathcal{O}\left((b_x - a_x) \sqrt{\frac{< f^2 > - < f >^2}{N}} \right)$$

where

$$\begin{aligned} < f > &= \frac{1}{N} \sum_{i=1}^{N} f(x_i) \\ < f^2 > &= \frac{1}{N} \sum_{i=1}^{N} f^2(x_i) \end{aligned}$$

Error estimate for other methods

Rectangle method

$$E = \mathcal{O}\left(\frac{(b_x - a_x)h}{2} f'\right) = \mathcal{O}\left(\frac{(b_x - a_x)^2}{2N} f'\right)$$

Trapezoidal method

$$E = \mathcal{O}\left(\frac{(b_x - a_x)h^2}{12} f''\right) = \mathcal{O}\left(\frac{(b_x - a_x)^3}{12N^2} f''\right)$$

Simpson method

$$E = \mathcal{O}\left(\frac{(b_x - a_x)h^4}{180} f^{(4)}\right) = \mathcal{O}\left(\frac{(b_x - a_x)^5}{180N^4} f^{(4)}\right)$$