

# Ordinary Differential equations

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Lecture 20

An ordinary equation of order  $n$  has the following form

$$y^{(n)} = f(x, y, y', y'', \dots, y^{(n-1)})$$

$x$  independent variable

$y^{(i)}$  the  $i_{th}$  derivative of  $y(x)$

$f$  often called the force term

# First order ODE example

## Example

the acceleration of the body is the first derivative of velocity with respect to the time and equals to the force divided by mass

$$a(t) = \frac{dv}{dt} = v'(t) = \frac{F}{m}$$

$t \rightarrow x$  independent variable

$v \rightarrow y$

$F/m \rightarrow f$

And we obtain the canonical form

$$y^{(1)} = f(x, y)$$

for the first order ODE

# $n_{th}$ order ODE transformation to the system of first order ODE

$$y^{(n)} = f(x, y, y', y'', \dots, y^{(n-1)})$$

we define the following variables

$$y_1 = y, y_2 = y', y_3 = y'', \dots, y_n = y^{(n-1)}$$

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \\ \vdots \\ y_{n-1}' \\ y_n' \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{n-1} \\ f_n \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ y_4 \\ \vdots \\ y_n \\ f(x, y_1, y_2, y_3, \dots, y_n) \end{pmatrix}$$

So we can rewrite  $n_{th}$  order ODE as a system of first order ODE

$$\vec{y}' = \vec{f}(x, \vec{y})$$

# System of the first order ODE and initial conditions

$$\vec{y}' = \vec{f}(x, \vec{y})$$

This is the system of  $n$  equations and thus requires  $n$  constraints.

Usually we specify  $\vec{y}(x_0) = \vec{y}_0$  i.e. initial conditions

$$\begin{pmatrix} y_1(x_0) \\ y_2(x_0) \\ y_3(x_0) \\ \vdots \\ y_n(x_0) \end{pmatrix} = \begin{pmatrix} y_{1_0} \\ y_{2_0} \\ y_{3_0} \\ \vdots \\ y_{n_0} \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \\ y_0'' \\ \vdots \\ y_0^{(n-1)} \end{pmatrix}$$

# Problem example

If acceleration of the particle is given and constant find the position as a function of time.

We are solving

$$x''(t) = a$$

we need to convert it to canonical form

$t \rightarrow x$  time as independent variable

$x \rightarrow y \rightarrow y_1$  particle position

$v \rightarrow y' \rightarrow y_2$  velocity

$a \rightarrow f$  acceleration as a force term

so

$$x'' = a \rightarrow y'' = f \rightarrow \vec{y}' = \vec{f}(x, \vec{y}) \rightarrow \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2 \\ f \end{pmatrix}$$

We also need initial conditions i.e. initial position  $x_0 \rightarrow y_{1_0}$  and velocity  $v_0 \rightarrow y_{2_0}$

# Euler's method

Let's for simplicity consider simple first order ODE (notice lack of vector)

$$y' = f(x, y)$$

There is an exact way to write the solution

$$y(x) = \int_{x_0}^x f(x, y) dx$$

The problem is that  $f(x, y)$  depends on  $y$  itself. However for small interval of  $x$ ,  $x + h$  we can assume that  $f(x, y)$  is constant

Then we get familiar box integration formula or in application to ODE the Euler's method.

$$y(x + h) - y(x) = \int_x^{x+h} f(x, y) dx \approx f(x, y)h$$

# Euler's method continued

$$y(x + h) = y(x) + f(x, y)h$$

All we need is to split our interval on bunch of steps of size  $h$ , and leap frog from the first  $x_0$  to the next one  $x_0 + h$ , then  $x_0 + 2h$  and so on. Now we can make an easy transformation to the vector case (i.e.  $n_{th}$  order ODE)

$$\vec{y}(x + h) = \vec{y}(x) + \vec{f}(x, y)h$$

Note: similarly to the boxes integration method  
**Euler's method is very imprecise for the given  $h$**



# Stability issue

Let's have a look at the first order ODE

$$y' = 3y - 4e^{-x}$$

It has the following solution

$$y = Ce^{3x} + e^{-x}$$

If our initial condition  $y(0) = 1$  the solution is  $y(x) = e^{-x}$ .

Please run [ode\\_unstable\\_example.m](#) and have a look at the output of the numerical solution

Clearly it's diverges from the analytical solutions. The problem is in round off errors which is the same as to say that  $y(0) = 1 + \delta$  then  $C \neq 0$  and solution diverges.

**Do not trust the numerical solutions (regardless of the method) without proper consideration!**