Numerical integration continued

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Lecture 11

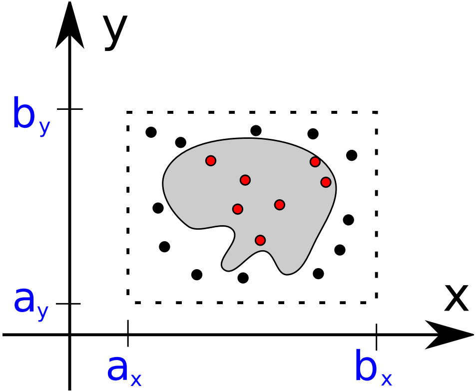
Midterm project

Due date Monday October 4th of 2010 at 1pm.

Discuss the relevant physics equation, describe you solution, show results. Matlab code might be left only for the email submission.

Problem (100 points total)
You are working for NASA. Your team is responsible to design a rocket which will lift off and after travel time $T_t = 50$ second in the gravity field of the Earth will reach certain orbit with final verical velocity $v_f = 0$. Do not worry about horizontal velocity. It is other team responsibility. Engineers provided you with an engine capable to provide to the rocket a time dependent lift acceleration in the form of
$a(t) = b * (1 - \exp(-t^2))$ (when other forces are disregarded) during time till fuel is cut off $T_c = 10$ second. The acceleration grows with time since rocket burns fuel and becomes lighter. However at time T_c no fuel is left and thus no lift force provided.

Toy example - area of the pond



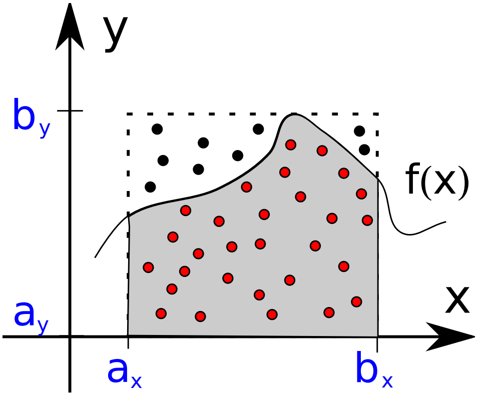
$$A_{pond} = \frac{N_{inside}}{N_{total}} A_{box}$$

where

$$A_{box} = (b_x - a_x)(b_y - a_y)$$

- Points must be uniformly and randomly distributed across the area.
- The smaller the enclosing box the better it is.

Naive Monte Carlo integration



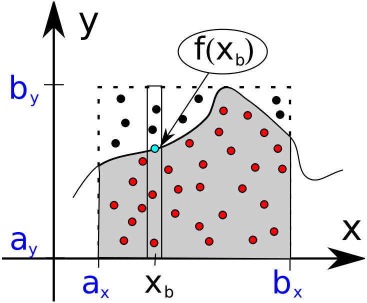
$$\int_{a_x}^{b_x} f(x)dx = \frac{N_{inside}}{N_{total}} A_{box}$$

where

$$A_{box} = (b_x - a_x)(b_y)$$

- Points must be uniformly and randomly distributed across the area.
- The smaller the enclosing box the better it is. So $max(f(x)) \rightarrow b_y$

Monte Carlo integration derived



Notice that if we choose a small stripe around the bin value x_b, then subset of points in that stripe gives an estimate for $f(x_b)$.
Thus why bother spreading points around area?

Let's chose a uniform random distribution of points x_i inside $[a_x, b_x]$

$$\int_{a_x}^{b_x} f(x)dx \approx \frac{b_x - a_x}{N} \sum_{i=1}^{N} f(x_i)$$

Error estimate for Monte-Carlo method

It can be shown that error of the numerical integration (E) is given by the following expressions

Monte Carlo method

$$E = \mathcal{O}\left((b_x - a_x)\sqrt{\frac{< f^2 > - < f >^2}{N}} \right)$$

where

$$< f > = \frac{1}{N}\sum_{i=1}^{N} f(x_i)$$

$$< f^2 > = \frac{1}{N}\sum_{i=1}^{N} f^2(x_i)$$

Error estimate for other methods

Rectangle method

$$E = \mathcal{O}\left(\frac{(b_x - a_x)h}{2} f'\right) = \mathcal{O}\left(\frac{(b_x - a_x)^2}{2N} f'\right)$$

Trapezoidal method

$$E = \mathcal{O}\left(\frac{(b_x - a_x)h^2}{12} f''\right) = \mathcal{O}\left(\frac{(b_x - a_x)^3}{12N^2} f''\right)$$

Simpson method

$$E = \mathcal{O}\left(\frac{(b_x - a_x)h^4}{180} f^{(4)}\right) = \mathcal{O}\left(\frac{(b_x - a_x)^5}{180N^4} f^{(4)}\right)$$