Root finding

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Lecture 06

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Example

$$3x^3 + 2 = \sin x \quad \rightarrow \quad 3x^3 + 2 - \sin x = 0$$

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A general search algorithm is the following

- make a guess i.e. trial
- make intelligent new guess (x_{i+1}) judging from this trial (x_i)
- continue until $|f(x_{i+1})| > \varepsilon_f$ and $|x_{i+1} - x_i| > \varepsilon_x$

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- some one think of any number between 1 and 100
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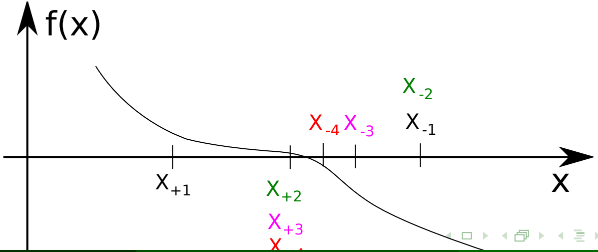
How many guesses do I need? At most 7

Bisection method pseudo code

Works for any continuous function in vicinity of function root

- make initial bracket for search x_+ and x_- such that
  - $f(x_+) > 0$
  - $f(x_-) < 0$
- loop begins
- make new guess value $x_g = (x_+ + x_-)/2$
- if $|f(x_g)| \leq \varepsilon_f$ or $|x_+ - x_g| \leq \varepsilon_x$
  stop we found the solution with desired approximation
- otherwise if $f(x_g) > 0$ then $x_+ = x_g$ else $x_- = x_g$
- continue the loop



Bisection - simplified matlab implementation

```
function x_sol=bisection(f, xn, xp, eps_f, eps_x)
% solving f(x)=0 with bisection method

  xg=(xp+xn)/2; % initial guess
  fg=f(xg);     % initial function evaluation

  while ( (abs(fg) > eps_f) & (abs(xg-xp)>eps_x) )
    if (fg>0)
      xp=xg;
    else
      xn=xg;
    end
    xg=(xp+xn)/2; % update guess
    fg=f(xg);     % update function evaluation
  end
  x_sol=xg; % solution is ready
end
```

Bisection - example of use

Let's define simple test function in the file 'function_to_solve.m'

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function ret=function_to_solve(x)
  ret=(x-10)*(x-20)*(x+3);
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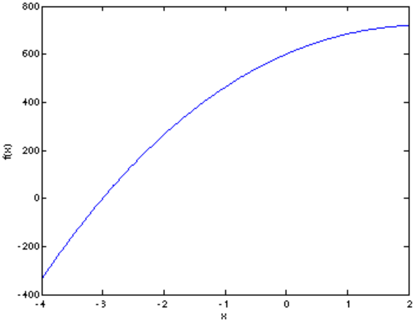

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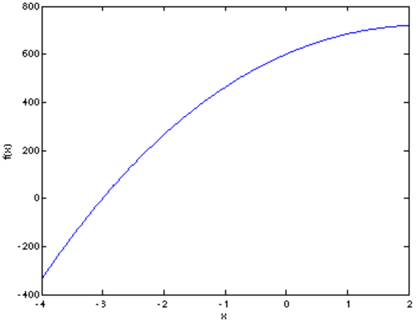


```
bisection(...
  @function_to_solve,...
  -4.1, 2, ...
  1e-6, 1e-8)
```

pay attention to the function handle operator @

```
ans = -3.0000
```

always cross check results

```
>> function_to_solve(ans)
ans = 3.0631e-07
```

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So what should we check for sure

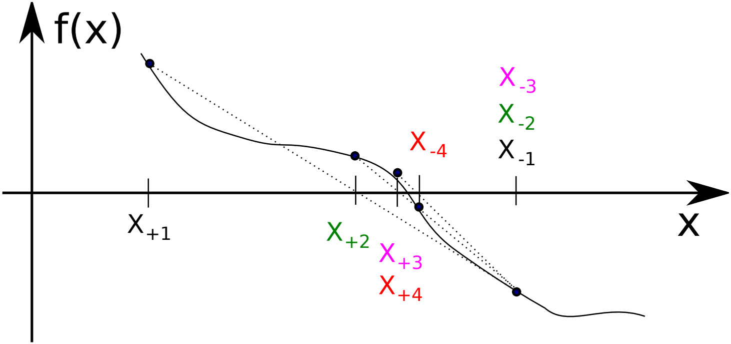
1. $f(xn) < 0$
2. $f(xp) > 0$

It would be handy to return secondary outputs

- with the value of function at the found solution point
- the number of iterations (good for performance tests)

False position (*regula falsi*) method

In this method we naively approximate our function as a line.



False position method - pseudo code

- make initial bracket for search x_+ and x_- such that
  - $f(x_+) > 0$
  - $f(x_-) < 0$
- loop begins
- draw a chord between points $(x_-, f(x_-))$ and $(x_+, f(x_+))$
- make new guess value at the point of the chord intersection with the 'x' axis
$$x_g = \frac{x_- f(x_+) - x_+ f(x_-)}{f(x_+) - f(x_-)}$$
- if $|f(x_g)| \le \varepsilon_f$ or $|x_+ - x_g| \le \varepsilon_x$
  stop we found the solution with desired approximation
- otherwise if $f(x_g) > 0$ then $x_+ = x_g$ else $x_- = x_g$
- continue the loop

Note: it looks like bisection except the way of updating x_g

Solution convergence

We say that algorithm has defined convergence if it is possible to express

$$\lim_{k\to\infty}(x_{k+1} - x_0) = c(x_k - x_0)^m$$

Where x_0 is true root of the equation, c is some constant, and m is the order of convergence.

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Generally the speed of the algorithm is related to its convergence order. How ever other factors may affect the speed.