

## Physics 786, Spring 2023

Problem Set 7, Due Friday, March 31.

### *Final Paper Assignment*

You should have a choice of topic by now, and you should be identifying relevant literature and reading up on your topic.

#### 1. *Vacuum Solutions in Three Dimensions*

In this problem we will look for static, isotropic black-holes in three space-time dimensions.

Assume a metric of the form

$$ds^2 = -e^{2\phi(r)} dt^2 + e^{2\lambda(r)} dr^2 + r^2 d\theta^2.$$

- a) Calculate the nonvanishing Christoffel symbols.
- b) Calculate the components of the Ricci tensor  $R_{tt}$ ,  $R_{rr}$ , and  $R_{\theta\theta}$ . The other components of  $R_{\mu\nu}$  vanish.
- c) Solve the vacuum Einstein equations  $R_{\mu\nu} = 0$ , assuming that the metric approaches the flat metric as  $r \rightarrow \infty$ . Are there any nontrivial black-hole solutions?

#### 2. *The Kerr spacetime*

*This problem is required for students registered for graduate credit, but is not required for students registered for undergraduate credit.*

In this problem you will consider the rotating black hole metric, discovered

by Roy Kerr in 1963. The metric can be written in Boyer-Lindquist form,

$$\begin{aligned}
 ds^2 = & - \left[ 1 - \frac{2GMr}{r^2 + a^2 \cos^2 \theta} \right] dt^2 - \frac{4GMr a \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} dt d\phi \\
 & + \left[ \frac{r^2 + a^2 \cos^2 \theta}{r^2 - 2GMr + a^2} \right] dr^2 + (r^2 + a^2 \cos^2 \theta) d\theta^2 \\
 & + \left[ r^2 + a^2 + \frac{2GMr a^2 \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} \right] \sin^2 \theta d\phi^2,
 \end{aligned}$$

where  $a$  is a constant related to the angular velocity of the black hole if  $M \neq 0$ .

If  $M = 0$  then the metric reduces to the Minkowski metric in oblate spheroid coordinates, related to usual Cartesian coordinates by

$$\begin{aligned}
 x &= \sqrt{r^2 + a^2} \sin \theta \cos \phi \\
 y &= \sqrt{r^2 + a^2} \sin \theta \sin \phi \\
 z &= r \cos \theta.
 \end{aligned}$$

a) Show that if  $a=0$  then the metric reduces to the Schwarzschild metric.

b) For what relations between  $r$  and  $\theta$  are components of the metric divergent? There should be several such relations.

c) The singularity at  $r = 0$  with  $\theta = \pi/2$  is a curvature singularity. The other metric singularities correspond to the horizons. What is the shape of the curvature singularity in terms of the coordinates  $x$ ,  $y$  and  $z$  as defined above?

d) Suppose an observer uses a jet pack to stand still, with constant  $r$ ,  $\theta$  and  $\phi$  in the Boyer-Lindquist coordinates. The tangent vector of the observer is  $v^\mu \equiv dX^\mu/dt = (1, 0, 0, 0)$ . This describes a physical tangent vector only if  $v^\mu$  is timelike, *i.e.*  $v^\mu v^\nu g_{\mu\nu} < 0$ . Find the relation(s) between  $r$  and  $\theta$  such that  $v^\mu v^\nu g_{\mu\nu} = 0$ .

These regions are called ergosurfaces. In the region (called the ergosphere) outside both horizons and inside the outer ergosurface, observers cannot stand still, but are dragged around the black hole. Roger Penrose suggested that energy could be taken from a rotating black hole by sending

objects through this region, with part of the objects falling into the black hole and the other parts propelled away from the black hole.

You should compare your results with Fig. 1 in Matt Visser's review at <https://arxiv.org/pdf/0706.0622.pdf>.