**Physics 786, Spring 2023**
**Problem Set 1** Due Wednesday, February 8, 2023.

1. *Lorentz tensors*

a) If $T^{\mu\nu}$ and $B_{\mu\nu}$ are tensors under Lorentz transformations, prove that $T^{\mu}{}_{\nu}B_{\mu}{}^{\nu}$ is a Lorentz scalar.

b) If $A^{\mu}(x)$ is a vector field, show that $\partial_\mu A_\nu(x)$ transforms like a (0,2) tensor under Lorentz transformations.

c) Write down all Lorentz scalars that contain exactly two factors of either $A^{\mu}(x)$ or its first derivatives.

d) Show that a symmetric (0,2) tensor $h_{\mu\nu} = h_{\nu\mu}$ remains symmetric following a Lorentz transformation.

e) If $h_{\mu\nu}(x)$ is a symmetric tensor field with $h_{\mu\nu}(x) = h_{\nu\mu}(x)$, write down all (and only) independent Lorentz scalars that contain two factors of either $h_{\mu\nu}(x)$ or its first derivatives.

f) Assume that the Minkowski metric, $\eta_{\mu\nu}$, transforms as a (0,2) tensor under Lorentz transformations. From the defining property of the Lorentz transformations, show that $\eta_{\mu\nu}$ is Lorentz invariant.

g) As in part (f), show that the Kronecker delta $\delta_{\mu}{}^{\nu}$ is an invariant tensor.

h) In units with $c = 1$, if $p^0 = E$ is the energy of a particle, and $\mathbf{p}$ is the three-vector momentum of the particle, identify the Lorentz invariant $p_\mu p^\mu$.

2. *The Levi-Civita symbol*

The Levi-Civita symbol $\epsilon^{\mu\nu\lambda\sigma}$ is antisymmetric under exchange of any two of its indices, with $\epsilon^{0123} = +1$. Show that as a tensor, $\epsilon^{\mu\nu\lambda\sigma}$ is invariant under Lorentz transformations with $\det\Lambda=+1$.

Note that the determinant of a $4{\times}4$ matrix $A$ with components $A_{\mu\nu}$, where

$\mu, \nu \in \{0, 1, 2, 3\}$, can be written

$$\det A = \sum_{\mu\nu\lambda\sigma} \epsilon^{\mu\nu\lambda\sigma} A_{0\mu} A_{1\nu} A_{2\lambda} A_{3\sigma}.$$