

weibing
12.4

An Action Principle for Gravity

The gravitational action should be invariant under coordinate transformations and should be composed of terms with two derivatives of the metric and its inverse.

The invariant volume element is $\sqrt{|g|} d^4x$, and the only scalar that suits the bill is R . Hence, we will study the equations that follow by varying the action $S = S_M + S_G$, with S_M the matter action, and

$$S_G = -\frac{1}{16\pi G} \int d^4x \sqrt{|g|} R$$

Taking $g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu}$, using $R = g^{\mu\nu} R_{\mu\nu}$,

$$\delta(\sqrt{|g|} R) = \sqrt{|g|} R_{\mu\nu} \delta g^{\mu\nu} + R \delta\sqrt{|g|} + \sqrt{|g|} g^{\mu\nu} \delta R_{\mu\nu}$$

From the definition of $R_{\mu\nu}$ in terms of $\Gamma_{\mu\nu}^\lambda$, and $\Gamma_{\mu\nu}^\lambda$ in terms of $g_{\mu\nu}$, it is straightforward to show the Palatini identity:

$$\delta R_{\mu\nu} = (\delta \Gamma_{m\lambda}^\lambda)_{;\nu} - (\delta \Gamma_{\mu\nu}^\lambda)_{;\lambda}$$

$$\text{Hence } \sqrt{|g|} g^{\mu\nu} \delta R_{\mu\nu} = \sqrt{|g|} \left[(g^{\mu\nu} \delta \Gamma_{m\lambda}^\lambda)_{;\nu} - (g^{\mu\nu} \delta \Gamma_{\mu\nu}^\lambda)_{;\lambda} \right]$$

$$\text{Using } V^{\mu}_{;\mu} = \frac{1}{\sqrt{|g|}} \partial_\mu (\sqrt{|g|} V^\mu),$$

$$\sqrt{|g|} g^{\mu\nu} \delta R_{\mu\nu} = \partial_\nu (\sqrt{|g|} g^{\mu\nu} \delta \Gamma_{\mu\lambda}^\lambda) - \partial_\lambda (\sqrt{|g|} g^{\mu\nu} \delta \Gamma_{\mu\nu}^\lambda)$$

Hence, $\int d^4x \sqrt{|g|} g^{\mu\nu} \delta R_{\mu\nu} = 0.$

Using $\delta \ln \det M = \text{Tr} M^{-1} \delta M$ with $M = g_{\mu\nu}$,

$$\delta \sqrt{|g|} = \frac{1}{2} \sqrt{|g|} g^{\mu\nu} \delta g_{\mu\nu}$$

Using $\delta (g^{\mu\nu} g_{\nu\lambda}) = 0$, $\delta g^{\mu\nu} = -g^{\mu\rho} g^{\nu\sigma} \delta g_{\rho\sigma}$

Hence, $\delta S_G = \frac{1}{16\pi G} \int d^4x \sqrt{|g|} \left[R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right] \delta g_{\mu\nu}$

The energy-momentum tensor for matter may be defined in terms of the variation of the matter action with respect to the metric:

$$\delta S_M = \frac{1}{2} \int d^4x \sqrt{|g|} T^{\mu\nu} \delta g_{\mu\nu}$$

Hence, $\delta S = \delta S_G + \delta S_M$
 $= \frac{1}{16\pi G} \int d^4x \sqrt{|g|} \delta g_{\mu\nu} \left[R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + 8\pi G T^{\mu\nu} \right]$

$\delta S = 0$ gives the equations of motion, which we recognize as the Einstein equations:

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = -8\pi G T^{\mu\nu}$$