VIII.1, Zee VIII.2, Weinberg 15

# The Expanding Universe

By modeling the universe as a perfect fluid we can use Einstein's eqs. to determine the evolution of the scale factor of the universe $R(t)$, and $K = +1, -1, \text{or } 0$.

$$ds^2 = -dt^2 + R^2(t)\left[\frac{dr^2}{1-Kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]$$

Nonvanishing components of the affine connection:

$$\Gamma^t_{rr} = \frac{R\dot{R}}{1-Kr^2}, \quad \Gamma^t_{\theta\theta} = R\dot{R}r^2, \quad \Gamma^t_{\varphi\varphi} = R\dot{R}r^2\sin^2\theta$$

$$\Gamma^r_{tr} = \Gamma^\theta_{t\theta} = \Gamma^\varphi_{t\varphi} = \frac{\dot{R}}{R}$$

Defining $g_{ij} \equiv R^2(t)\,\tilde{g}_{ij}$ for $i,j \in \{r,\theta,\varphi\}$

$$\Gamma^i_{jk} = \frac{1}{2}(\tilde{g}^{-1})^{il}\left(\frac{\partial \tilde{g}_{lj}}{\partial x^k} + \frac{\partial \tilde{g}_{lk}}{\partial x^j} - \frac{\partial \tilde{g}_{jk}}{\partial x^l}\right) \equiv \tilde{\Gamma}^i_{jk}$$

Determined from 3D metric $\tilde{g}_{ij}$

Ricci tensor: $R_{tt} = \dfrac{3\ddot{R}}{R}$

$$R_{ti} = 0$$

$$R_{ij} = \tilde{R}_{ij} - (R\ddot{R} + 2\dot{R}^2)\,\tilde{g}_{ij}$$

where

$$\tilde{R}_{ij} = \partial_j \tilde{\Gamma}^k_{ki} - \partial_k \tilde{\Gamma}^k_{ij} + \tilde{\Gamma}^k_{li}\tilde{\Gamma}^l_{kj} - \tilde{\Gamma}^k_{ij}\tilde{\Gamma}^l_{kl}$$
$$= -2K\,\tilde{g}_{ij}$$

$$\Rightarrow R_{ij} = -\left(R\ddot{R} + 2\dot{R}^2 + 2k\right)\tilde{g}_{ij}$$

Einstein Eqs: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G\, T_{\mu\nu}$

Trace: $-R^\lambda_\lambda = -8\pi G\, T^\lambda_\lambda$

$$\Rightarrow \boxed{R_{\mu\nu} = -8\pi G\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\lambda_\lambda\right)}$$

$$\equiv -8\pi G\, S_{\mu\nu}$$

where $S_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T$

Perfect Fluid: $T_{\mu\nu} = p\, g_{\mu\nu} + (p+\rho) U_\mu U_\nu$

where $U^\mu = (1,0,0,0)$ — fluid elements are at rest in comoving coordinates.

Assume $p, \rho$ depend only on $t$ (homogeneous, isotropic)

$$S_{\mu\nu} = \frac{1}{2}(\rho - p) g_{\mu\nu} + (p+\rho) U_\mu U_\nu$$

$$\Rightarrow \boxed{\begin{aligned} S_{tt} &= \tfrac{1}{2}(\rho + 3p) \\ S_{it} &= 0 \\ S_{ij} &= \tfrac{1}{2}(\rho - p) R^2 \tilde{g}_{ij} \end{aligned}}$$

$$R_{tt} = -8\pi G\, S_{tt} \Rightarrow \boxed{3\ddot{R} = -4\pi G(\rho + 3p) R}$$

$$R_{ij} = -8\pi G\, S_{ij} \Rightarrow \boxed{R\ddot{R} + 2\dot{R}^2 + 2K = 4\pi G(\rho - p) R^2}$$

$$\Rightarrow \frac{1}{2} R_{ij} - \frac{1}{6} R\, R_{tt} = \boxed{\dot{R}^2 + K = \frac{8\pi G}{3} \rho R^2}$$

scale factor not curvature scalar

Conservation of Energy-Momentum:

$$0 = D_\nu T^{\mu\nu}$$

$$= \partial_\nu p g^{\mu\nu} + \frac{1}{\sqrt{g}} \partial_\nu \left[\sqrt{g}\,(\rho+p)\, U^\mu U^\nu\right] + \Gamma^\mu_{\nu\lambda} (\rho+p) U^\nu U^\lambda$$

$$\mu = t: \quad -\dot{p} + \frac{\cancel{\sqrt{1-kr^2}}}{\cancel{r^2 \sin\theta}\, R^3} \frac{\cancel{r^2 \sin\theta}}{\cancel{\sqrt{1-kr^2}}} \frac{d}{dt}\left[R^3(\rho+p)\right] + 0 = 0$$

$$\Rightarrow \dot{p} R^3 = \frac{d}{dt}\left[R^3(\rho+p)\right]$$

$$-p \frac{d}{dt}(R^3) = \frac{d}{dt}(R^3 \rho)$$

$$= -p \cdot 3R^2 \frac{dR}{dt}$$

$$\Rightarrow \boxed{\frac{d}{dR}(\rho R^3) = -3pR^2}$$

Given eqn. of state $\boxed{p = p(\rho)}$, this equation determines $\rho$ as a function of $R$.

<u>Example</u>: pressureless dust $p = 0$ } Matter-dominated (after radiation domination)

$$\Rightarrow \rho \propto R^{-3}$$

<u>Example</u>: Massless photons $p = \frac{\rho}{3}$ } Radiation-Dominated (Early universe)

$$\rho \propto R^{-4}$$

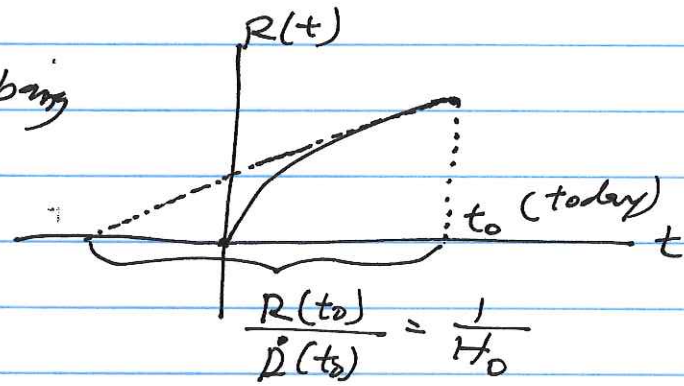
tt component of Einstein's equations:

$$3\ddot{R} = -4\pi G(\rho+3p)R$$

$\Rightarrow$ As long as $(\rho+3p)>0$, $\frac{\ddot{R}}{R}<0$

$\rightarrow$ Expansion rate of the universe <u>decelerates</u> in this model.

$t=0$: Big bang




"Age of universe" $t_0 < \frac{1}{H_0}$ in this model.

$$D_\nu T^{\mu\nu}=0 \rightarrow \frac{d}{dR}(\rho R^3) = -3pR^2$$

$$p>0 \rightarrow \frac{d}{dR}(\rho R^3) < 0$$

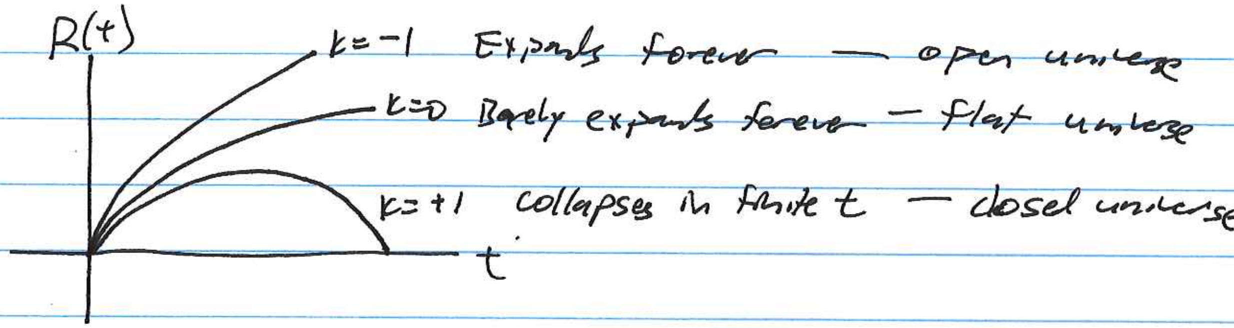
$\Rightarrow \rho$ decreases with $R$ at least as fast as $R^{-3}$.

Large $R$: $\dot{R}^2+K = \frac{8\pi G}{3}\rho R^2 \rightarrow 0$

$K=-1$: $\dot{R}^2 \rightarrow 1$ at $\infty$: $R(t)\rightarrow t$ as $t\rightarrow\infty$

$K=0$: $\dot{R}^2\rightarrow 0$ at $\infty$

$K=+1$: $\dot{R}^2 \rightarrow 0$ at $\rho R^2 = \frac{3}{8\pi G}$, then $\dot{R}<0$ for larger $R$




k=0 is the borderline case between an ever-expanding universe and an eventually collapsing universe in this model.

$$\dot{R}^2 + k = \frac{8\pi G}{3}\rho R^2$$

$$k=0 \Leftrightarrow \rho = \frac{3}{8\pi G}\frac{\dot{R}^2}{R^2} = \boxed{\frac{3H^2}{8\pi G} \equiv \rho_c}$$ Critical density

$$k=+1 \Leftrightarrow \rho > \rho_c$$
$$k=-1 \Leftrightarrow \rho < \rho_c$$

Matter-Dominated Universe: $\rho = \rho_0 \frac{R_0^3}{R^3}$

$$\dot{R}^2 + k = \underbrace{\frac{8\pi G}{3}\rho_0 R_0^3}_{C}\frac{R^2}{R^3} = \frac{C}{R}$$

$$k=+1: \quad \dot{R}^2 + 1 = \frac{C}{R} = \frac{R_{max}}{R} \qquad (\dot{R}=0 \rightarrow C = R_{max})$$

Solution: $$\begin{cases} R = \frac{C}{2}(1-\cos\eta) = \frac{R_{max}}{2}(1-\cos\eta) \\ t = \frac{R_{max}}{2}(\eta - \sin\eta) \end{cases}, \quad 0 \le \eta \le 2\pi$$

Total lifetime of Universe: $R=0 \rightarrow \eta = 2\pi$

$$\rightarrow \boxed{t = \pi R_{max}}$$

$K=-1$: $\dot{R}^2 - 1 = \frac{C}{R}$

Solution: $\begin{cases} R = \frac{C}{2}(\cosh\eta - 1) \\ t = \frac{C}{2}(\sinh\eta - \eta) \end{cases}$ , $\eta > 0$

Large $t$: $R \sim t$

Small $R$: $\dot{R}^2 \sim \frac{C}{R}$ $\Rightarrow \sqrt{R}\, dR \sim C^{1/2} dt$

$\frac{2}{3} R^{3/2} \sim \sqrt{C}\, t$

$R \sim \left(\frac{3}{2}\right)^{2/3} C^{1/2} t^{2/3}$

Small-$R$ behavior independent of $K$.

$K=0$: $R = \left(\frac{3}{2}\right)^{2/3} C^{1/2} t^{2/3}$

where $C = \frac{8\pi G}{3} \rho_0 R_0^3$

## The Multicomponent Universe

We will consider a more general universe containing several perfect fluids with equations of state approximately given by $p_i = w_i \rho_i$, where $w_i$ is called the <u>equation of state parameter</u> of the fluid component labeled by $i$.

There is a separate conservation equation for each fluid component:

$$\frac{d}{dR}(\rho_i R^3) = -3 p_i R^2,$$ which can be written

$$\frac{d\rho_i}{dR} + 3(p_i + \rho_i)\cdot\frac{1}{R} = 0$$

Multiplying by $\frac{dR}{dt}$,

$$\boxed{\frac{d\rho_i}{dt} + 3(p_i + \rho_i)H = 0},$$ where $H = \frac{\dot{R}}{R}$.

With $p_i = w_i \rho_i$, $\quad \frac{d\rho_i}{dt} + 3(1+w_i)\rho_i \frac{\dot{R}}{R} = 0$

The redshift parameter is $\boxed{z = \frac{R_0}{R} - 1}$

$$\frac{dz}{dt} = -\frac{R_0}{R^2}\dot{R}$$

$$\frac{d\rho_i}{dt}\cdot\frac{dt}{dz} + 3(1+w_i)\rho_i \frac{\dot{R}}{R}\left(-\frac{R^2}{R_0 \dot{R}}\right) = 0$$

$$\frac{d\rho_i}{dz} - 3(1+w_i)\rho_i (1+z)^{-1} = 0$$

Solution: $\boxed{\rho_i = \rho_{0i}(1+z)^{3(1+w_i)}}$

The Friedmann Equations can be written

$$\boxed{H^2 = \frac{8\pi G}{3}\sum_i \rho_i - \frac{k}{R^2}}$$

Defining $\Omega_i \equiv \frac{\rho_i}{\rho_c} = \frac{8\pi G \rho_i}{3H^2}$, $\Omega_{curv} \equiv -\frac{k}{R^2H^2}$,

the Friedmann eqn. takes the form

$$\boxed{\sum_i \Omega_i + \Omega_{curv} = 1}$$

Given the solution for $\rho_i(z)$,

$$\boxed{H^2(z) = H_0^{\,2}(1+z)^2\Big(1+\sum_i \underbrace{\Omega_i}_{\text{today}}\big[(1+z)^{1+3w_i} - 1\big]\Big)}$$

This formula for the Hubble constant is useful for constraining the properties of the matter content of the universe in terms of $\Omega_i$ and $w_i$.

Imagine a distant light source with absolute luminosity $L$ (energy/time). The light is detected at $r=0$ with flux $F$ (energy/time/area) and redshift $z_s$. ← source

The proper area of a sphere at coordinate radius $r_s$ is $4\pi r_s^2 R_0^2$.

The energy of each emitted photon is reduced by the factor $(1+z_s)^{-1}$, and the time interval between photons received is increased by another factor of $(1+z_s)^{-1}$.

Hence, $\boxed{L = 4\pi r_s^2 R_0^2 (1+z_s)^2 F}$

The luminosity distance $d_L$ is defined so that

$L = 4\pi d_L^2 F$, so that

$\boxed{d_L^2 = r_s^2 R_0^2 (1+z_s)^2}$ in an FRW universe.

Measurement of F and $z_s$ with knowledge of L allows for a determination of $r_s(z)$, or alternatively $d_L(z)$. Such a measurement can be compared with the prediction for $r_s(z)$ given a set of $\Omega_i, w_i$:

The light travels along a trajectory with $ds^2 = 0$

$$\rightarrow \int_0^{r_s} \frac{dr}{\sqrt{1-kr^2}} = \int_{t_s}^{t_0} \frac{dt}{R(t)} = \int_0^{z_s} \frac{dz}{R_0 H(z)}$$

For example, assuming $k=0$,

$$\boxed{r_s(z_s) = \frac{1}{R_0 H_0} \int_0^{z_s} \frac{dz}{(1+z)\sqrt{1 + \sum_i \Omega_i \left[(1+z)^{1+3w_i} - 1\right]}}}$$

Type Ia supernovae have known intrinsic luminosities, so they are a good tool for observing the Hubble curve, which is often given as magnitude vs. redshift, where the magnitude m of a bright object is defined by

$$m = \underbrace{m_0}_{\text{fiducial source magnitude}} + 5\log_{10}(H_0 d_L)$$